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THE SOLAR PROMINENCE OF JUNE 19–20, 1911

By FREDERICK SLOCUM

On June 19 and 20, 1911, I observed on the northwest limb of the sun a prominence which possessed some especially interesting features. Twenty-five photographs in the light of the H-line of calcium were made with the Rumford spectroheliograph of the Yerkes Observatory on the 19th, between 8 A.M. and 4 P.M., of which twenty-one are reproduced in Plates XV, XVI, and XVII. On the 20th, nine exposures were made, of which six are reproduced in Plate XVII.

The sky remained clear and the seeing good throughout both days, so that a fairly complete record of the changes in the prominence was secured. The exposure time, that is, the time required for the image of the prominence to pass over the slit, was thirteen seconds, and the photographs shown are separated by intervals ranging from 1.7 to 100 minutes.

On the 19th, the prominence maintained approximately the same position and dimensions, while undergoing numerous striking internal changes. Its height remained about $2'\frac{1}{4}$, or 97,000 km. With the exception of a short time between 8^h and $8^h\ 30^m$ G.M.T., its extent along the limb was between $4'$ and $5'$, or between 172,000 km and 215,000 km. It extended from latitude $+35^\circ$ to latitude $+49^\circ$ on the west limb.

When first observed, the prominence showed three conspicuous features; the arch on the left, the pillar in the center, and the lofty extension on the right.

The arch, at first relatively thin, increased rapidly in thickness, while the space under the arch diminished in area. Up to 4^h $33^m.8$, this space appeared intensely black and the line of demarkation between the bright arch and the interior space was sharp. This is especially well shown at 4^h $1^m.5$. After that time, the space under the arch began to fill up, and the arch itself became less massive, and finally, at about 8^h, completely disappeared. At 8^h $58^m.0$, a new thin arch appeared, and this gradually increased in thickness until, at 9^h $47^m.5$, it resembled quite closely the original arch observed seven hours earlier.

At 5^h $32^m.9$ and 5^h $34^m.9$, interesting spiral columns appear, as if supporting the arch. Between 8^h $58^m.0$ and 9^h $45^m.5$, the second arch changed from a curved to an angular, and then back to a curved form.

The central pillar started, at 2^h $18^m.2$, with an anvil-like form, 45,000 km high. This gradually increased in height to a maximum of 65,000 km, at 4^h $36^m.9$, its apparent motion, therefore, being at the rate of 2.4 km per second. Throughout its development it was curving to the right or south. After 4^h $36^m.9$, this pillar began to disintegrate, and, at 7^h $39^m.5$, it was replaced by a form somewhat resembling a sky-rocket with a large and brilliant head. This head moved to the south at the rate of 13 km per second, and was soon lost in the higher portion of the prominence. At 8^h $17^m.5$, the pillar had again assumed an anvil shape, but, at 8^h $58^m.0$, it was replaced by a form resembling a tornado cloud, and at 9^h $55^m.3$ two of these vortices appeared.

Considering, now, the third feature, we find that on the first plate it resembled a chimney from which smoke is issuing, and this smoke appears to be driven to the left or north by some lateral current. For several hours the base of the chimney became more and more massive and the volume of smoke increased up to 3^h $12^m.5$, then decreased rapidly, but enough remained to indicate throughout the day the existence of an upper current from the south. That this is merely an upper current is shown by the pho-

tographs of Plate XV, in which the forms of intermediate height, that is, up to about 60,000 km, are streaming in the opposite direction.

On the next day, the upper current was reversed. At $2^h\ 3^m.3$, June 20, the prominence resembled two volcanoes in eruption. The smaller one quickly disappeared, and by $4^h\ 58^m.1$, the larger one had nearly vanished. This larger volcano-like form, extending from latitude 41° to latitude 44°, was the chief source of activity on this date, but sporadic eruptions appeared for some distance on either side of it. Note, for example, the eruption on the right of the photograph taken at $3^h\ 9^m.5$ (Plate XVII). The arch in this figure is almost a perfect semicircle, 25,000 km in diameter, nearly twice the diameter of the earth. No trace of it appears on the plate taken eleven minutes earlier, and, on a plate exposed 2.6 minutes later, only the remnants of it are visible.

After passing through many transformations, the prominence finally became dissipated and floated away upward. This method of dissolution is characteristic of by far the greater number of the solar prominences whose last stages I have observed.

Yerkes Observatory
April 17, 1912

THE PRESSURE DISPLACEMENT OF SPECTRAL LINES

By T. H. HAVELOCK

The object of the following note is to suggest a comparison between observed pressure effects in emission spectra and a displacement of absorption lines which can be deduced theoretically from a certain type of dispersion formula. This is introduced by a preliminary discussion of existing theories, which deal with the direct effect upon emission. Although the calculations necessarily refer to ideally simple cases, they may be tested by seeing if they give effects of the right order of magnitude and if they are in general agreement with experimental results.

CRITICAL STUDY OF CERTAIN THEORIES

1. Humphreys[1] ascribes the effect to the mutual interaction of atomic magnetic fields and thus makes the pressure-shift comparable with the Zeeman effect. In estimating the order of magnitude, Humphreys considers iron atoms in the electric arc at atmospheric pressure, using among other data: radius of atom, 10^{-8} cm; average distance between centers of atoms, $6 \cdot 10^{-7}$ cm. Thus the calculation assumes the vapor-density to be that of a perfect gas at a pressure of one atmosphere and at 2730° absolute, having $4 . 10^{18}$ atoms per cubic centimeter. In order to obtain a displacement of a line of the right amount, it is necessary for the atomic magnetic fields to be enormously high, and this involves the assumption of a very large number of electrons revolving in circular orbits within the atom. The calculation has been criticized by Richardson,[2] who shows that with a more reasonable estimate of the number of revolving electrons the atomic magnetic fields are such that their mutual influence is entirely negligible; the present writer agrees with this conclusion. The comparison of experimental results for the Zeeman effect and the pressure-

[1] *Astrophysical Journal*, **23**, 233, 1906.

[2] *Philosophical Magazine*, **14**, 557, 1907. See also Sanford, *Astrophysical Journal*, **35**, 1, 1912.

shift is discussed later. We may note here that the above calculation appears to make the pressure displacement depend upon the density of the emitting metallic vapor, this in turn corresponding to the total pressure. For if a is the average distance between atoms, the magnetic field due to a neighboring atom is approximately proportional to a^{-3}; but a^{-3} is proportional to the number, N, of atoms per unit volume, and so the effect is a linear function of the pressure. It should be added that in later papers, Humphreys considers the change of period of a metallic atom in the arc to be due to the magnetic fields of the atoms of the gas in which the arc is burning. There appears to be no direct evidence that the effect depends upon magnetic properties, and in any case we decide that these are insufficient to give effects of the right magnitude.

2. We consider now some calculations which involve the mutual influence of the electric fields of neighboring atoms. It was suggested by Fitzgerald that the change in the specific inductive capacity K of the surrounding gas due to change of pressure might account for the variation of the period of a vibrating atom in the arc. We may in fact assume the period of the vibrator to be proportional to the square root of K. In some estimates from this point of view, for instance by Humphreys,[1] the resulting displacement comes out many times too large; but the cause of this lies in treating the surrounding gas as a continuous medium extending right up to the particular vibrator in question. This error appears to be at the root of various statements that the pressure-shift cannot be due to the mutual influence of atomic electric fields, because calculations based thereon give far too large a result.

Larmor,[2] on the other hand, takes into account the molecular constitution of the gas. Considering the period of a spherical vibrator of radius a, which behaves like a simple Hertzian doublet, Larmor replaces the surrounding gas by a uniform medium, of inductive capacity K, extending up to a distance ka from the center of the vibrator; thus the latter is at the center of a spherical

[1] *Astrophysical Journal*, **26**, 30, 1907.

[2] *Ibid.*, 120, 1907.

cavity in a continuous medium. For the displacement $d\lambda$ of a vibration of wave-length λ, due to an increase of pressure of one atmosphere, the calculation gives an approximate formula

$$\frac{d\lambda}{\lambda}=\frac{1}{2k^3}\,\frac{K-1}{K} \qquad (1)$$

Taking K as 1.0006 for air at a pressure of one atmosphere, and assuming 10^{-6} as an average observed value of $d\lambda/\lambda$, this gives k equal to about 7. Larmor concludes that although the data for molecular magnitudes are of course vague, they appear to justify the statement that the dielectric influence of the neighboring molecules is a *vera causa* of the right order of magnitude.

This conclusion may be criticized on the following grounds. The above value of K is that for air at a temperature of 273° absolute. Although the temperature conditions of the arc are uncertain, it seems more reasonable to put the temperature of the immediate atmosphere of gas at some conventional value like 2730° absolute, as in the calculation in the previous section. At this temperature, $K-1$ must be taken as $6\cdot 10^{-5}$, one-tenth of the value above. The formula (1) now gives k equal to 3. But the molecules at this temperature are spaced roughly at 60 times the molecular radius; so that the value of k necessary for the right result seems inadmissible. We conclude that the dielectric influence of the neighboring molecules of the *surrounding gas* leads to a displacement many times too small.

Further, the formula (1) applies primarily no doubt to a simple illustration, but it indicates certain relations. For instance, the displacement should vary with the value of $K-1$ for the gas; but this has not been confirmed by experiment. In addition, k^{-3} is roughly proportional to density, and we have made $K-1$ also proportional to density in the above calculation. It follows that the displacement should vary as the square of the pressure; but the relation obtained experimentally is a linear one.

3. Another calculation on the same general lines is that of Richardson (*loc. cit.*). The argument is that, if a metallic atom A is emitting radiation, its frequency is affected by the sympathetic electric vibrations induced in the atoms B of the surrounding gas. Richardson obtains greater detail in his formula by definitely

assuming the vibrators to be electronic charges, as in ordinary optical theory; the result is

$$\frac{d\lambda}{\lambda}=\frac{e^2\lambda^2(\mu^2-1)}{6\pi^2mc^2\alpha^3} \qquad (2)$$

where μ is the refractive index of the surrounding gas. The quantity α is taken as the radius of the sphere within which it is impossible for the center of an atom of class B to lie; the value of α is supposed to lie within the limits a and $2a$, where a is the atomic radius. Taking a mean value $1.5\cdot10^{-8}$ cm for α, Richardson calculates the order of magnitude for a line of wave-length $4\cdot10^{-5}$ cm. The surrounding gas is taken to be air at 2730° absolute, for which μ^2-1 is $5.9\cdot10^{-5}$. With the usual values of e, m, and c, the formula (2) gives per unit atmosphere a proportional displacement $d\lambda/\lambda$ equal to $9\cdot10^{-5}$; that is, the result is about 100 times an average observed result such as we have used in the previous sections.

The error appears to be one to which reference has already been made. The effect of atoms of class B is obtained as an integral with α as its lower limit, and with the value of α chosen, the surrounding gas is made equivalent to a continuous medium extending right up to the vibrating atom A; but this is not permissible. If for a moment we replace α by ka as in Larmor's calculation and find k from (2) so as to give the right order of magnitude for $d\lambda$, we find k equal to 7; this seems too small, considering the condition of the gas with its atoms spaced roughly at 50 or 60 times the atomic radius. Further, if we have to make α variable with the density in this way, we encounter again the difficulty that the formula would make the displacement proportional to the square of the pressure, instead of the first power. Also as before, the factor μ^2-1 does not accord with experimental results.

4. We notice that in Richardson's theory, as in Larmor's, the calculations refer entirely to the dielectric influence of the surrounding gas, and we conclude that this gives a displacement which is many times too small. The calculations ignore the mutual electric influence of neighboring atoms of the metallic vapor, that is, of atoms having the same free periods. Richardson, in fact, expressly rules out of consideration the effect of an increase in the partial

pressure of the metallic vapor, stating that this produces in general only a broadening of the lines without displacement. This may or may not be the case, but it does not follow from the theoretical considerations; for these might be made to apply to a radiating atom surrounded by similar atoms, with a resulting displacement many times larger owing to resonance effects. In fact the influence of similar atoms could be ignored only if the vapor-density were very small compared with that of an ideal gas under similar conditions. This point of view has been stated by Schuster;[1] after giving an account of Larmor's theory, he writes: "The question is complicated by the fact that in the cases which have been observed, the greater portion of the metallic vapor vibrates in an atmosphere of similar molecules." K of formula (1) is then replaced by μ^2, where μ is the refractive index of the vapor. One plan is to put μ^2 infinite for a free period; this gives $d\lambda/\lambda$ equal to $\frac{1}{2}k^{-3}$, and with k equal to 10, the displacement comes out about 500 times too large. Schuster remarks that this method gives no exact information about the displacement of the position of maximum intensity. In addition, it is unsatisfactory in that Larmor's formula assumed K to be not much different from unity. However, it may be noticed that with k equal to 80 we get an effect of the right order; this supports the contention that the effect of neighboring similar atoms can be ignored in this connection only if the vapor-density is very small.

If we attempt to make a more complete theory on these lines, we meet all the difficulties of the emission of an aggregate of molecules. The previous theories consider the radiation of a single particle, but what is required is the period for maximum intensity of emission of a collection of similar vibrators. One knows that similar difficulties have arisen in the theory of the direct Zeeman effect, with the result that writers have turned to the inverse effect; it has been found easier to obtain more detail from the consideration of absorption spectra, chiefly because one can work from known theories of dispersion. A similar step is suggested now in the theory of the pressure displacement of spectral lines.[2]

[1] *Encyclopaedia Brittanica*, eleventh edition, **25**, 628.

[2] See also *Proceedings of the Royal Society*, A, **84**, 517, 1911.

THE DISPLACEMENT OF ABSORPTION MAXIMA

5. If n is the refractive index of a homogeneous medium for waves of frequency $p/2\pi$, we have a dispersion formula

$$\frac{n^2-1}{n^2+2}=\sum\frac{\frac{4\pi}{3}N\frac{e^2}{m}}{p_1^2-p^2} \tag{3}$$

We shall find it convenient to use the ordinary interpretation of the constants. There are N vibrators per unit volume, of mass m, carrying a charge e, and having a natural frequency p_1; the summation extends over all the types of vibrators. This formula includes the effect of the electric polarization of neighboring particles when estimating the electric force acting on a vibrator; in the present connection it may be as well to repeat briefly the argument.

Let d be the displacement at any time of a typical particle vibrating about a position of equilibrium. Let E be the electric intensity and P the total polarization of the medium. The particle is supposed to be acted on by a field $E+\frac{4\pi}{3}P$, as if it were in a spherical cavity in a medium uniformly polarized at each instant to the value P; among other conditions necessary, it must be possible to surround each point by a sphere which contains a large number of particles but whose radius is small compared with the wave-length of the radiation. The equation of motion of the particle is

$$m\frac{\delta^2 d}{\delta t^2}=-mp_1^2d+e\left(E+\frac{4\pi}{3}P\right) \tag{4}$$

Hence for radiation of frequency p,

$$Ned=\frac{Ne^2/m}{p_1^2-p^2}\left(E+\frac{4\pi}{3}P\right)$$

But the total polarization P is ΣNed, taken over all types of vibrator. Combining these results with the fact that P equals $(n^2-1)E/4\pi$, the dispersion formula (3) is deduced. We may write the formula as

$$\frac{n^2-1}{n^2+2}=\frac{n'^2-1}{n'^2+2}+\frac{\frac{4\pi}{3}N\frac{e^2}{m}}{p_1^2-p^2} \tag{5}$$

In the term in n' are included all the terms of the summation except that in p_1. We may call n' the refractive index of the medium if all the vibrators of natural frequency p_1 were destroyed without disturbing the rest of the medium. In most cases the maximum absorption caused by this type of vibrator occurs very near to p_1, so we may treat n' as constant in the following argument. The frequency p_1' for maximum absorption is found to a first approximation by making n infinite in (5); we obtain

$$p_1'^2 = p_1^2 - \frac{n'^2+2}{3} \cdot \frac{4\pi}{3} N \frac{e^2}{m} \tag{6}$$

It appears that the squares of the frequencies p_1 and p_1' differ by a quantity proportional to the effective density of vibrators of the particular type. In general the displacement due to the last term in (6) is very small relatively; in terms of wave-length we have

$$\frac{d\lambda_1}{\lambda_1^3} = \frac{n'^2+2}{3} \cdot \frac{Ne^2}{6\pi^2 mc^2} \tag{7}$$

where the displacement $d\lambda_1$, equal to $\lambda_1' - \lambda_1$, is measured from the wave-length corresponding to the natural vibrations of an isolated particle. The formula follows naturally from the dispersion formula; it remains to be seen how it agrees with observed pressure effects.

6. In the first place, we can simplify the formula by noticing that the factor $\frac{1}{3}(n'^2+2)$ is very nearly equal to unity for gaseous media. In particular, this factor may be taken to represent the influence of the surrounding gas. For if we have N vibrators of frequency p_1 per unit volume uniformly disposed in two different media, the displacements will be in the ratio of the values of $\frac{1}{3}(n^2+2)$ for the two media.

In the theories discussed in the previous sections, the surrounding gas supplies a factor n^2-1. If this were the case the effect should be observable; for instance, for an arc under pressure in air and in carbon dioxide, the displacements should be in the ratio of 2 to 3. Rossi[1] concludes from his experiments that in this case the displacements are the same within the limits of experimental error. This result would be expected from formula

[1] *Philosophical Magazine*, **21**, 499, 1911.

(7); for an estimate, we may assume n' to be the refractive index of the gas at 50 atmospheres and from (7) the ratio of the displacements in air and carbon dioxide is 1 to 1.005, other things being equal. We conclude that the effect due to change of refractive index of the surrounding gas is inappreciable. The determining factor according to the present argument is the density N of similar vibrators, and it is conceivable that the surrounding gas might have some direct effect on the value of N.

7. We simplify the formula by putting unity for n', so that

$$\frac{d\lambda}{\lambda^3}=\frac{Ne^2}{6\pi c^2 m} \tag{8}$$

We have no direct evidence as to the value of N, so we cannot prove directly that (8) gives a displacement of the right order of magnitude. But by using an observed value of $d\lambda$ we can calculate N and see if it is a reasonable amount. We take the data from recent experiments by Gale and Adams.[1]

For a group of iron lines of average wave-length 4287 Å, the displacement $d\lambda$ per atmosphere is 0.00274 Å. We express these in centimeters and put in the formula; we also substitute the usual values of the electronic charge and mass. In this way we obtain from this example

$$N=2.3\times 10^{16} \tag{9}$$

If the vapor were an ideal gas at 2730° absolute, a pressure of one atmosphere would correspond to a value of about 4×10^{18} for N. But the temperature and other conditions in the arc are too uncertain to allow of an estimate of the vapor-density; in addition, as in similar investigations, it is probable that only a certain fraction of the metallic atoms is concerned in the production of a given line in the spectrum. We conclude that the values of N deduced from (8) are not unreasonable; at least they do not prohibit further consideration of the application of the formula to pressure effects.

8. The displacement of any line depends upon the effective density of similar vibrators; this is no doubt proportional to the vapor-density, but it probably varies from one line to another,

[1] *Astrophysical Journal*, **35**, 10, 1912.

as may be inferred from the values of the corresponding coefficients in dispersion formulae. Therefore one cannot expect, from (8), any general law of variation of displacement with wave-length; but other things being equal, it indicates proportionality to the cube of the wave-length. This relation has been extracted from experimental results by Duffield in the case of gold, and for iron by Gale and Adams. The process consists in taking averages for large numbers of lines in groups; one may suppose that, if there is no definite relation between N and λ, this process gets rid of the chance variations of N and exhibits the displacement varying as λ^3. On the other hand, for connected lines for which N is a function of λ, the displacement will follow some other law. In certain cases the various lines can be separated roughly into groups for which the values of (displacement)/(wave-length)^3 are nearly in simple ratios such as 1 : 2 : 3. In terms of the formula (8), one might suppose that N is of the same order of magnitude for all the lines and then the simple ratios would be associated with the values of e/m for the vibrators.

9. It is of interest to compare this with a simple theory of the Zeeman effect. If H is the external magnetic field, it can be shown that the displacement, in frequency, is $eH/4\pi m$; or in wave-lengths

$$\frac{d\lambda}{\lambda^2}=\frac{eH}{4\pi mc} \tag{10}$$

In this case the number of similar vibrators is not brought into consideration, since the action is regarded as a direct effect of the external field upon each vibrator. The formula is simpler, and it has been found easier to analyze results by means of it, than for the pressure effect. Experimental results have been compared carefully in order to discover any possible connection between the Zeeman separation and the pressure displacement, but with little result; the most that can be said in certain cases is that, taking the means of separation and displacement for large numbers of lines in the same region of the spectrum, it is found that these means are of the same order of magnitude when they are classified broadly as small, medium, and large.[1]

[1] A. S. King, *Astrophysical Journal*, **31**, 433, 1910; also *ibid.*, **34**, 250, 1911.

From the present point of view no connection is to be expected in general; for the formula for the pressure effect makes $d\lambda/\lambda^3$ proportional to Ne^2/m, and N varies in an unknown manner with the wave-length. If the conditions are such that N is of the same order for groups of lines, one might expect then some correspondence between the values of $d\lambda/\lambda^2$ for the Zeeman effect and $d\lambda/\lambda^3$ for the pressure effect. In other words, any correspondence that may be extracted from experimental results is sufficiently accounted for by the fact that both effects are due to a slight disturbance of the same vibrating system; it is not necessary to suppose that the disturbances are produced in the same way.

10. A conspicuous effect of pressure is a broadening of the lines. To discuss this, we should have to introduce absorption terms into the dispersion formulae. The simplest suppositions give, instead of (3),

$$\left.\begin{aligned} n^2(1-\kappa^2) &= 1+4\pi N\frac{e^2}{m}\,\frac{p'^2_1-p^2}{(p'^2_1-p^2)^2+b_1^2p^2} \\ 2n^2\kappa &= 4\pi N\frac{e^2}{m}\,\frac{b_1 p}{(p_1^2-p^2)^2+b_1^2p^2} \end{aligned}\right\} \quad (11)$$

The maximum of $n\kappa$ occurs at a frequency slightly different from p'_1; the difference involves the damping coefficient b_1 and is relatively very small in most cases. The maximum value of $n\kappa$ is proportional to $Ne^2/mb_1p'_1$. Roughly speaking, a greater value of b_1 means a broader and weaker line; while an increase in N both strengthens and broadens the absorption. One might try to connect the displacement of a line with its intensity, but for the occurrence of the factor b_1; whatever be the physical mechanism of absorption, b_1 probably varies from line to line and also with the physical conditions. Also, experimentally one has only estimates of relative intensities at atmospheric pressure and it is known that these vary with the pressure. A further complication is the occurrence of reversals under experimental conditions; so on the whole it is useless to follow this line farther at present.

Some writers have argued that the pressure effect cannot be due to increased vapor-density, or the proximity of similar molecules, because it is known that mere increase of emitting vapor in the arc intensifies and broadens lines without producing any observed

displacement of the maximum. If this is the case, it may be due to an increase in the number of radiators without any considerable change in the physical conditions of the aggregate; just as in an absorption spectrum an increased thickness of the absorbing medium broadens and intensifies each region of absorption without appreciably displacing the maximum. Or again, it may be due to a change in the absorption represented by a change in the damping coefficient b_1.

11. The arguments which have been put forward may be summarized briefly.

Certain theories of the pressure effect are discussed, and it is concluded that they all lead to a displacement which is many times too small; these theories ascribe the effect to the magnetic influence of neighboring atoms of the metallic vapor or of the surrounding gas, or to the dielectric influence of the surrounding gas.

It is suggested that an effect of the right order may be obtained from the electric influence of neighboring similar atoms of the metallic vapor. The formula relies on a known deduction from a dispersion formula and it applies primarily to absorption lines. Considerations are advanced to show that the formula bears comparison in general with experimental results.

Armstrong College
Newcastle-on-Tyne
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ON THE DETERMINATION OF THE ORBITAL ELEMENTS OF ECLIPSING VARIABLE STARS. I

By HENRY NORRIS RUSSELL

§ 1. *Statement of the problem.*—Bauschinger, in his exhaustive work on the determination of orbits, remarks concerning the problem of determining the elements of the orbit and the dimensions and brightness of the component stars of an eclipsing variable from the observed light-curve:[1] "Der Zusammenhang zwischen den Grössen-, Formen- und Helligkeitsverhältnissen der Körper und den Elementen der elliptischen Bahn einerseits und der Lichtkurve anderseits ist aber ein so komplizierter, dass man eine allgemeine Theorie wohl kaum aufstellen kann, sondern die Lösung von Fall zu Fall den vorliegenden Verhältnissen anpassen muss."

It is the purpose of the present discussion to show under what circumstances, and to what degree, this problem may be regarded as determinate (in view of the limited accuracy of photometric observations), and to develop formulae and tables which make the solution of the problem, when it is determinate, a simple matter.

In the most general case, the number of unknown quantities to be determined is considerable. The relative orbit will in general be eccentric, and the two components of the system unequal in size and brightness. They may present the appearance of disks not uniformly illuminated, but darkened toward the limb, and may also be elongated toward one another by their mutual attraction, and brighter on the side receiving the radiation of the companion than on that remote from it.

For a complete specification of such a system we must therefore know at least 13 quantities, as follows:

Orbital Elements		Eclipse Elements	
Semi-major axis	a	Radius of larger star	r_1
Eccentricity	e	Radius of smaller star	r_2
Longitude of periastron	ω	Light of larger star	L_1

[1] *Die Bahnbestimmung der Himmelskörper* (Leipzig, 1906), p. 649.

Orbital Elements		Eclipse Elements	
Inclination	i	Light of smaller star	L_2
Period	P	and at least 3 constants defining the amount of elongation, of darkening at the limb, and of brightening of one star by the radiation of the other.	
Epoch of principal conjunction	t_0		

The longitude of the node must remain unknown, as there is no hope of telescopic separation of any eclipsing pair.

The value of a in absolute units can be found only from spectroscopic data. In the absence of these it is desirable to take a as an unknown but definite unit of length, and express all other linear dimensions in terms of it. Similarly, the absolute values of L_1 and L_2 can be determined only if the parallax of the system is known. But in all cases the combined light of the pair, L_1+L_2, can be taken as the unit of light and the apparent brightness at any time expressed in terms of this. This leaves the problem with eleven unknown quantities to be determined from the photometric measures. Of these, the period is invariably known with a degree of accuracy greatly surpassing that attainable for any of the other elements, and the epoch of principal minimum can be determined, almost independently of the other elements, by inspection of the light-curve. Of the remaining elements, the constants expressing ellipticity and "reflection" may be derived from the observed brightness between eclipses.[1] These effects are often so small as to be detected only by the most refined observations.[2] The question of darkening toward the limb may well be set aside until the problem is solved for the case of stars which appear as uniformly illuminated disks.

This leaves us with six unknowns. Fortunately, systems of such short period as the majority of eclipsing variables have usually nearly circular orbits (as is shown both by spectroscopic data and by the position of the secondary minimum). The assumption of a circular orbit is therefore usually a good approximation to the facts, and often requires no subsequent modification.

[1] E. C. Pickering, *Proceedings of American Academy of Arts and Sciences*, **16**, 257, 1880.

[2] R. S. Dugan, *Contributions from the Princeton University Observatory*, No. 1, p. 37, 1911; J. Stebbins, *Astrophysical Journal*, **32**, 200, 1910.

PART I. SOLUTION FOR SPHERICAL STARS AND CIRCULAR ORBIT

We will therefore first discuss the following simplified problem: *Two spherical stars, appearing as uniformly illuminated disks, and revolving about their common center of gravity in circular orbits, mutually eclipse one another. It is required to find the relative dimensions and brightness of the two stars, and the inclination of the orbit, from the observed light-curve.* The questions arising out of orbital eccentricity, ellipticity, "reflection," and darkening toward the limb will be discussed later.

§ 2. *Notation. Possible cases.*—In this simplified problem we may assume P and t_0 as already known. If the radius of the relative orbit is taken as the unit of length, and the combined light of the two stars as the unit of light, we have to determine four unknown quantities. Of the various possible sets of unknowns, we will select the following:

Radius of the larger star	r_1
Ratio of radii of the two stars	k
Light of the larger star	L_1
Inclination of the orbit	i

The radius of the smaller star is then $r_2 = kr_1$, and its light, $L_2 = 1 - L_1$. It should be noticed that, with the above definitions, k can never exceed unity, but L_2 will exceed L_1 whenever the smaller star is the brighter (which seems to be the fact in the majority of observed cases).

We will suppose that we have at our disposal a well-determined "light-curve," or more accurately, magnitude-curve, giving the stellar magnitude m, which we will suppose to be the vertical co-ordinate, as a function of the time, as horizontal co-ordinate. From this we can pass at once to the intensity-curve, giving the actual light-intensity l as a function of the time, by means of the equation

$$\log l = 0.4(m_0 - m), \tag{1}$$

where m_0 is the magnitude during the intervals of constant light between eclipses (which is determined with relatively great weight by the observations during these periods and, like P, may be found once for all before beginning the real solution). This of course expresses l in terms of our chosen unit $L_1 + L_2$.

Such a magnitude-curve or intensity-curve will in general show two depressions, or "minima," corresponding to the mutual eclipses of the two components. Under the assumed conditions, it is well known:

1. That the two minima will be symmetrical about their middle points, and that these will be separated by exactly half the period.

2. If the eclipse is total or annular, there will be a constant phase at minimum during which the magnitude- or intensity-curve is horizontal; but if the eclipse is only partial, this will not be the case.

3. The two minima will be of equal duration, but usually of unequal depth. At any given phase during one minimum one of the stars will eclipse a certain area of the apparent disk of the other. Exactly half a period later, at the corresponding phase during the other minimum, the geometrical relations of the two projected disks will be the same, except that now the second star is in front, and eclipses an equal area—though not an equal proportion—of the disk of the first. The intensity-curves for the two minima must therefore differ from one another only as regards their vertical scales, which will be in the ratio of the surface intensities of the two stars.

4. The deeper (primary) minimum corresponds to the eclipse of the star which has the greater surface intensity by the other. Whether this is the larger or smaller star must be determined by further investigation.

Suppose that at any time during the eclipse of the smaller star by the larger the fraction α of its area is hidden. The light received from the system at this moment will be given by the equation

$$l_1 = 1 - \alpha L_2. \tag{2}$$

Half a period later, an equal area of the surface of the larger disk, and hence the fraction $k^2\alpha$ of its whole area will be eclipsed. The observed light will then be

$$l_2 = 1 - k^2 \alpha L_1. \tag{3}$$

Since $L_1 + L_2 = 1$, we find at once from these equations

$$(1 - l_1) + \frac{1 - l_2}{k^2} = \alpha. \tag{4}$$

It should be especially noticed that the subscript 1 in these equations applies to the eclipse of the *small* star by the *large* one, and *not* necessarily to the principal minimum.

We may now distinguish four cases of our problem, according to the form of the light-curve, and the extent of our knowledge of the secondary minimum.

1. Both primary and secondary minima have been observed, and show a constant phase.

2. The primary minimum shows a constant phase, and the secondary has not been adequately observed.

3. Both minima have been observed, but show no constant phase.

4. Only the primary minimum, showing no constant phase, has been observed.

In the first case the eclipse of the smaller star by the larger is total, and the other annular. In both cases during the constant phase the whole area of the smaller star is obscured; that is, $a=1$. If then λ_1 and λ_2 are the values of the observed intensity during these constant phases, we have, by (4),

$$k^2=\frac{1-\lambda_2}{\lambda_1}. \tag{5}$$

Moreover, by (2), $\lambda_1=1-L_2=L_1$. The brightness of the two stars and the ratio of their radii, are thus determined, leaving only r_1 and i to be found.

There are, however, two solutions with different values of k according as we regard the principal or secondary minimum as total. We shall see later how we may distinguish the correct solution in a given instance.

In the second case, if the observed minimum intensity is λ and we assume that the observed eclipse is total, we have from (2), $L_2=1-\lambda$; if annular, (3) gives $k^2L_1=1-\lambda$. In either case, for any other value l of the observed intensity,

$$a=\frac{1-l}{1-\lambda}. \tag{6}$$

We thus know a as a function of the time, and from this have to determine k, r_1, and i.

In the third case, if λ_1 and λ_2 are the observed intensities at the minimum phases of the two eclipses, and a_0 is the corresponding value of a, we have

$$a_0 = 1 - \lambda_1 + \frac{1-\lambda_2}{k^2}. \tag{7}$$

Since $a_0 = \frac{1-\lambda_1}{L_2}$, we may take it as an unknown instead of L_1 or L_2. We then have to find the four quantities, r_1, i, k, and a_0 with the aid of the intensity-curve and the equation (7).

In the fourth case the situation is the same except that the equation (7) is not available.

§ 3. *Solution when the eclipse is total.*—The solutions in the other cases may be derived from that in Case II, which will now be developed.

Take the center of the larger star as origin, and let θ be the true longitude of the smaller star in its orbit, measured from inferior conjunction. Then

$$\theta = \frac{2\pi}{P}(t - t_0). \tag{8}$$

From the light-curve and (6) we can find the value of a for any value of θ, or vice versa. Now a, which is the fraction of the area of the smaller disk which is eclipsed at any time, depends on the radii of the two disks, and the apparent distance of their centers, but only on the ratios of these quantities (being unaffected by increasing all three in the same proportion). If δ is the apparent distance of centers, we have therefore

$$a = f\left(\frac{r_2}{r_1}, \frac{\delta}{r_1}\right) = f\left(k, \frac{\delta}{r_1}\right),$$

where f is a function, the details of calculation of which will be discussed later.

For any given value of k we may invert this function, and write

$$\frac{\delta}{r_1} = \phi(k, a). \tag{9}$$

This function, or some equivalent one, may be tabulated once for all for suitable intervals of k and a, as is done in Table I below,

which gives a function $p(k, a)$ such that $\phi(k, a) = 1 + kp(k, a)$. By the geometry of the system, we have

$$\delta^2 = \sin^2 \theta + \cos^2 i \cos^2 \theta = \cos^2 i + \sin^2 i \sin^2 \theta, \qquad (10)$$

whence

$$\cos^2 i + \sin^2 i \sin^2 \theta = r_1^2 \{\phi(k, a)\}^2. \qquad (11)$$

Now let a_1, a_2, a_3 be any definite values of a and $\theta_1, \theta_2, \theta_3$ the corresponding values of θ (which may be found from the light-curve). Subtracting the corresponding equations of the form (11) in pairs, and dividing one of the resulting equations by the other, we find

$$\frac{\sin^2 \theta_1 - \sin^2 \theta_2}{\sin^2 \theta_2 - \sin^2 \theta_3} = \frac{\{\phi(k, a_1)\}^2 - \{\phi(k, a_2)\}^2}{\{\phi(k, a_2)\}^2 - \{\phi(k, a_3)\}^2} = \psi(k, a_1, a_2, a_3). \qquad (12)$$

The first member of this equation contains only known quantities. The second, if a_1, a_2, and a_3 are predetermined, is a function of k alone. If this function is tabulated, the value of k in any given case can be found by interpolation, or graphically. The equations (11) can then be used to find r_1 and i.

A theoretical light-curve may then be found, which passes through any three desired points on each branch of the observed curve (assumed symmetrical). These points may be chosen at will by altering the values of a_1, a_2, and a_3. In practice it is convenient to keep a_2 and a_3 fixed, so that ψ becomes a function of k and a_1 only, and may be tabulated for suitable intervals in these two arguments. This has been done in Table II, in which a_2 is taken as 0.6 and a_3 as 0.9. If $A = \sin^2 \theta_2$, $B = \sin^2 \theta_2 - \sin^2 \theta_3$ (12) may be written

$$\sin^2 \theta_1 = A + B\psi(k, a_1). \qquad (13)$$

The points a and b on the light-curve corresponding to a_2 and a_3, together with the point corresponding to any one of the tabular values of a_1, then give a determination of k. By taking a suitably weighted mean of these values of k, a theoretical light-curve can be obtained which passes through the points a and b, and as close as possible to the others. By slight changes in the assumed positions of a and b (i.e., in the corresponding values of θ, or of $t - t_0$), it is possible with little labor to obtain a theoretical curve which

fits the whole course of the observed curve almost as well as one determined by least squares. The criterion of this is that the parts of the observed curve below b (near totality), between a and b, and above a (near the beginning or end of eclipse) shall give the same mean value of k. The individual determinations of k are of very different weight. Between a and b (that is for values of α_1 between 0.6 and 0.9) ψ changes very slowly with k. At the beginning and end of the eclipse the stellar magnitude changes very slowly with the time, and hence, by (13), with ψ. The corresponding parts of the curve are therefore ill adapted to determine k. For the first approximation it is well to give the values of k derived from values of α_1 between 0.95 and 0.99, and between 0.4 and 0.2, double weight (provided the corresponding parts of the curve are well fixed by observation). The time of beginning or end of eclipse cannot be read with even approximate accuracy from the observed curve and should not be used at all in finding k. The beginning or end of totality may sometimes be determined with fair precision, but does not deserve as much weight as the neighboring points on the steep part of the curve. If further refinement is desired, it can most easily be obtained by plotting the light-curve for two values of k and comparing with a plot of the observations. This will rarely be necessary.

When once k is given, the determination of the light-curve is a very easy matter. For each tabular value of α_1, equation (13) gives θ_1, and hence t_1-t_0. The values of the stellar magnitude m corresponding to given values of α_1 are already available, having been used in the previous work. The light-curve may thus be plotted by points in a few minutes.

After a satisfactory light-curve has been computed, we may proceed to determine the remaining elements. Let θ' and θ'' be the values corresponding to the beginning of eclipse ($\alpha_1=0$) and to the beginning of totality ($\alpha_1=1$). Then by (13)

$$\sin^2\theta'=A+B\psi(k, 0) \text{ and } \sin^2\theta''=A+B\psi(k, 1).$$

These computed values are more accurate than those estimated from the free-hand curve drawn to represent the observations.

At the first of these epochs $\delta = r_1 + r_2$, and at the second $\delta = r_1 - r_2$. We have then, by (10)

$$r_1^2(1+k)^2 = \cos^2 i + \sin^2 i \sin^2 \theta',$$
$$r_1^2(1-k)^2 = \cos^2 i + \sin^2 i \sin^2 \theta'',$$

whence

$$4k \cot^2 i = (1-k)^2 \sin^2 \theta' - (1+k)^2 \sin^2 \theta'',$$
$$4kr_1^2(1+\cot^2 i) = \sin^2 \theta' - \sin^2 \theta''.$$

Introducing A and B, we have

$$4k \cot^2 i = -4kA + B\{(1-k)^2\psi(k, 0) - (1+k)^2\psi(k, 1)\},$$
$$4kr_1^2 \operatorname{cosec}^2 i = B\{\psi(k, 0) - \psi(k, 1)\}.$$

The coefficients are functions of k alone, and may be tabulated. It is most convenient for this purpose to put the equations in the form

$$\left.\begin{aligned} r_1^2 \operatorname{cosec}^2 i &= \frac{B}{\phi_1(k)}, \\ \cot^2 i &= \frac{B}{\phi_2(k)} - A, \end{aligned}\right\} \qquad (14)$$

as in this way we obtain functions whose tabular differences are comparatively smooth (which is not true of their reciprocals). With the aid of these functions the elements may be found as soon as A and B are known. If $\frac{B}{A} < \phi_2(k)$ the computed value of $\cot i$ is imaginary and the solution is physically impossible. It is therefore advisable to apply this test to the preliminary values of A, B, and k, and, if necessary, to adjust them so that the solution is real. The limiting condition is evidently $\cot i = 0$, corresponding to central transit.

The geometrical elements of the system are now determined. We are still in doubt, however, whether the principal eclipse is total or annular. This can be determined only by consideration of the secondary minimum. The intensities during constant phase from eq (5) at the two minima are connected by the relation $k^2\lambda_1 + \lambda_2 = 1$. If the intensity at principal minimum is λ_p, that at the secondary minimum will be $1 - k^2\lambda_p$ if the principal eclipse is total, and $\frac{1-\lambda_p}{k^2}$

if it is annular. The first of these expressions is always positive and less than unity. The second exceeds unity if $1-\lambda_p>k^2$. The assumption of total eclipse at principal minimum leads therefore in all cases to a physically possible solution. That of an annular eclipse does so only if $1-\lambda_p$ is not greater than k^2. Otherwise the computed brightness of the smaller star is negative. The brightness at secondary minimum will be greater than at the primary by $1-\lambda_p(1+k^2)$ if the primary eclipse is total, and $\frac{1}{k^2}\{1-\lambda_p(1+k^2)\}$ if it is annular. The latter hypothesis therefore gives rise to the shallower minimum. In many cases it may be impossible to decide between the two without actual observations of the secondary phase. The computed depth of secondary minimum may, however, be so great that it is practically certain that it would sometimes have been observed if it really existed. The corresponding hypothesis should then be rejected. If $\lambda_p(1+k^2)$ is nearly unity, the primary and secondary minimum, on both hypotheses, must be of nearly equal depth. This can occur only if $\lambda_p<\frac{1}{2}$; that is, if the depth of minimum is less than 0.75 mag. In such a case it is probable that the period is really twice that so far assumed, that the two stars are of equal surface brightness, and that two sensibly equal eclipses occur during each revolution. The true values of θ are therefore half those previously computed with the shorter period. If the determination of k is repeated on this basis, and the equation $\lambda_p(1+k^2)=1$ is still approximately satisfied this solution may be adopted.

Such a system presents a specialized example of Case I, when both primary and secondary minima have been observed and show a constant phase. In this case, by (5), $k^2=\frac{1-\lambda_2}{\lambda_1}$ where λ_1 corresponds to the total eclipse, which, so far as we yet know, may occur at either minimum. As before we begin by finding from the light-curve the values of $\sin^2\theta$ corresponding to given values of α_1. From a few of these, by the method already described, an approximate value of k may be obtained which is sufficient to show which of the values given by (5) on the two possible hypotheses is the correct one.

We have next to find the light-curve which gives the best representation of the observations consistent with the value of k given by (5). The form of the light-curve now depends only on the constants A and B in the equation

$$\sin^2 \theta_1 = A + B\psi(k, \alpha_1). \tag{13}$$

Approximate values of these constants may be derived as above from the values of $\sin^2 \theta$ when $\alpha = 0.6$ and 0.9. These may be improved by trial and error, which will be aided by plotting the resulting light-curves along with the observations, and, if the data warrant it, may finally be corrected by least squares.[1] When satisfactory values of A and B have been determined, the final light-curve may be computed by (13), and the elements by (14), as in Case II, except that here there is no uncertainty as regards the nature of the principal eclipse.

In review of the foregoing, it may be remarked that the method of solution is direct and simple. It involves a very moderate amount of numerical work, of which the greater part—namely, the determination of the values of the magnitude, time, and position in orbit (θ) corresponding to different percentages of obscuration (α)—requires no modification during the successive approximations. The light-curve may be found without the necessity of

[1] The equations of condition are of the form

$$\frac{dm}{d(\sin^2 \theta)}\delta A + \psi(k, \alpha)\frac{dm}{d(\sin^2 \theta)}\delta B = \text{O.} - \text{C.}$$

where the second member represents the difference between the observed and computed magnitudes. The coefficients may easily be obtained graphically. By plotting the computed magnitude m against $\sin^2 \theta$, the values of $\frac{dm}{d(\sin^2 \theta)}$ may be read off, from the slope of the tangent, at each point used in the construction of the curve. The values of ψ and of $t - t_0$ corresponding to these points are already known, and hence the coefficients of the equations (of condition) may be plotted as functions of $t - t_0$, and read off for each observation.

A similar process might be adopted when only the primary minimum has been observed, including a correction to k among the unknowns. The equations of condition would then be of the form

$$\frac{dm}{d(\sin^2 \theta)}\left(\delta A + \psi(k, \alpha)\delta B + B\frac{\delta\psi}{\delta k}\delta k\right) = \text{O.} - \text{C.}$$

This would, however, be worth while only when computation showed that the secondary minimum must be so shallow as to be practically unobservable, as otherwise elements derived from the primary minimum alone could in no sense be regarded as definitive.

computing the elements, and with two or three trials may be determined so as to represent the whole course of the observations, making the laborious solution by least squares superfluous except in the case of observations of unusual precision. Such a solution itself is much simplified if the constants defining the light-curve, instead of the elements of the system, are treated as the fundamental unknowns, as the coefficients of the equations of condition may be easily found graphically with the aid of data already computed. The elements may be found, at any stage of the process, by a few moments' calculation, from the constants defining the light-curve.

§ 4. *Solution when the eclipse is partial.*—Passing now to Cases III and IV, where the observed curve shows no constant phase, and the eclipse is partial, we can no longer use the previous methods since the values of α corresponding to the maximum eclipse, or to any other phase, are unknown. We may, however, determine the magnitude corresponding to an obscuration of any given fraction of the maximum obscuration α_0, and the corresponding values of l and θ.

Let n represent this fraction. We then have

$$\alpha = n\alpha_0\,, \quad \text{and} \quad 1-l=n(1-\lambda). \tag{15}$$

Let $\theta(n)$ denote the corresponding value of θ. Then we have by (13)

$$\sin^2\theta(n)=A+B\psi(k,\ n\alpha_0).$$

The value $n=1$ corresponds to the middle of eclipse when $\theta=0$. Hence

$$A+B\psi(k,\ \alpha_0)=0.$$

Subtracting, we have

$$\sin^2\theta(n)=B\{\psi(k,\ n\alpha_0)-\psi(k,\ \alpha_0)\}.$$

Dividing this by the similar equation for any fixed value of n (say $\frac{1}{2}$), we eliminate the constants of the individual light-curve, and find

$$\frac{\sin^2\theta(n)}{\sin^2\theta(\frac{1}{2})}=\frac{\psi(k,\ n\alpha_0)-\psi(k,\ \alpha_0)}{\psi(k,\ \frac{1}{2}\alpha_0)-\psi(k,\ \alpha_0)}=\chi(k,\ \alpha_0,\ n), \tag{16}$$

(where the above equation is to be taken as the definition of the new function χ). The first member contains only known quantities. The second is a function of k, α_0, and n, which may be

tabulated for any convenient values. We might then expect to solve the problem by constructing two or more such tables (e.g., for $n=\frac{1}{4}$ and $n=\frac{3}{4}$), and finding for what values of k and a_0 the two functions $\chi(k, a_0, \frac{1}{4})$ and $\chi(k, a_0, \frac{3}{4})$ had the values assigned each by equation (16).

But when this experiment is actually tried it is found that the functions χ (regarded as a function of k and a_0 for different constant values of n) are all so nearly functions of one another that the solution becomes practically indeterminate. In other words, if we give k and a_0 any pair of values which make some one of these functions, say $\chi(k, a_0, \frac{1}{4})$ equal to a given value, we will by this very process constrain all the other functions $\chi(k, a_0, n)$ to be very nearly (though not exactly) equal to certain other constant values (depending of course on n).

This may be illustrated by the following examples which give, for given values of this function, pairs of corresponding values of k and a_0, and the resulting values of two other functions of the series.

$\chi(k, a_0, \frac{1}{4})$	k	a_0	$\chi(k, a_0, \frac{3}{4})$	$\chi(k, a_0, 0)$
2.20	0.89	1.00	0.32	5.05
	1.00	0.91	.32	5.08
2.00	1.00	0.74	0.37	4.24
	0.90	.84	.37	4.26
	.80	.98	.38	4.26
1.80	1.00	0.45	0.42	3.41
	0.90	.52	.43	3.42
	.80	.60	.43	3.42
	.70	.73	.42	3.45
	.62	.90	.43	3.45
	.64	1.00	.44	3.46
1.70	1.00	0.22	0.45	3.03
	0.80	.27	.46	3.01
	.60	.42	.45	3.04
	.50	.55	.46	3.02
	.48	.70	.46	3.02
	.48	.90	.45	3.07
	.53	1.00	.48	3.11
1.60	0.00	0.28	0.48	2.57
	.10	.48	.48	2.58
	.14	.60	.49	2.57
	.23	.80	.49	2.62
	.41	1.00	.52	2.72

The time of beginning of eclipse cannot be determined from the curve with any certainty, and hence $\chi(k, \alpha_0, 0)$ is valueless for determining k and α_0. It is clear that to obtain a reliable determination of them in this way, it would be necessary to carry the function $\chi(k, \alpha_0, \frac{3}{4})$ and the ratios $\frac{\sin^2 \theta(\frac{3}{4})}{\sin^2 \theta(\frac{1}{2})}$ to at least three decimal places; that is, that we should be able to determine the interval during which the star is apparently below a given magnitude during eclipse to within one part in a thousand. This is obviously out of the question. We may therefore conclude: *To the degree of approximation attained by any existing photometric measures, the problem of determining the elements of an eclipsing variable solely from the light-curve of a primary minimum without constant phase is indeterminate.* For any value k of the ratio of the radii of the components, comprised within wide limits, it is possible to find an assumed percentage α_0 of maximum eclipse, and hence a set of elements, such that the interval from the middle of eclipse, at which any given magnitude is reached, as calculated from any of these systems of elements, will be the same within a fraction of 1 per cent.

For practical purposes, therefore, we may regard the functions $\chi(k, \alpha_0, n)$ as functions of n and of any one function of the set, e.g., $\chi(k, \alpha_0, \frac{1}{4})$. By a happy accident, the relation between any pair of them, corresponding to different fixed values of n, is very nearly linear, as is illustrated below.

$\chi(k, \alpha_0, \frac{1}{4})$	$\chi(k, \alpha_0, 0)$	Comp.	O.−C.	$\chi(k, \alpha_0, \frac{3}{4})$	Comp.	O.−C.
2.46	6.15	6.15	0.00	0.238	0.235	+0.003
2.40	5.90	5.90	.00	.249	.253	− .004
2.30	5.50	5.49	+ .01	.282	.283	− .001
2.20	5.08	5.08	.00	.310	.313	− .003
2.10	4.68	4.67	+ .01	.345	.343	+ .002
2.00	4.25	4.26	− .01	.378	.373	+ .005
1.90	3.84	3.85	− .01	.407	.403	+ .004
1.80	3.44	3.44	.00	.433	.433	.000
1.70	3.04	3.03	+ .01	.458	.463	− .005
1.60	2.61	2.62	− .01	.490	.493	− .003
1.50	2.32	2.21	+ .11	.540	.523	+ .017
1.41	2.00	1.84	+ .16	.590	.550	+ .040

The tabular values of $\chi(k, \alpha_0, 0)$ and $\chi(k, \alpha_0, \frac{3}{4})$ are the means of the slightly varying values (such as are given in the preceding

table) for the different values of k and a_0, which make $\chi(k, a_0, \frac{1}{4})$ equal to the value given in the first column. The columns headed "Comp." are derived from the equations:

$$\chi(k, a_0, 0) = 4.10\chi(k, a_0, \tfrac{1}{4}) - 3.94$$
$$\chi(k, a_0, \tfrac{3}{4}) = 0.973 - 0.300\chi(k, a_0, \tfrac{1}{4}).$$

These formulae, although wholly empirical, represent the computed data almost within the errors of reckoning, except for values of $\chi(k, a_0, \frac{1}{4})$ less than 1.60. These last can occur only for values of k and a_0 so small that they are very unlikely to be met with in practice, so that the linear formulae are abundantly sufficient in all ordinary cases.

Similar linear relations are fulfilled with an equal degree of approximation for other values of n—the coefficients being functions of this quantity. We may therefore write in general

$$\chi(k, a_0, n) = \omega_1(n) + \omega_2(n)\chi(k, a_0, \tfrac{1}{4}). \tag{17}$$

The three functions occurring in this expression have been computed and tabulated, and may be considered known. But by (16) we have

$$\sin^2 \theta(n) = \sin^2 \theta(\tfrac{1}{2})\chi(k, a_0, n),$$

whence

$$\sin^2 \theta(n) = \omega_1(n) \sin^2 \theta(\tfrac{1}{2}) + \omega_2(n) \sin^2 \theta(\tfrac{1}{4}).$$

If we set $\sin^2 \theta(\frac{1}{4}) = C$, $\sin^2 \theta(\frac{1}{2}) = D$, this becomes

$$\sin^2 \theta(n) = C\omega_2(n) + D\omega_1(n). \tag{18}$$

This equation, though not rigorously exact, must be very nearly satisfied in the case of all light-curves arising from partial eclipses. Having given any observed light-curve, the values of C and D may be determined in the manner already described on page 325, and a theoretical light-curve be found, which closely represents the whole course of the observations.

Knowledge of this light-curve, however, does not enable us to find the elements, but only a relation between them of the form

$$\chi(k, a_0, \tfrac{1}{4}) = \frac{C}{D}. \tag{19}$$

Unless the secondary minimum has been observed, we can go no farther. But if we know the brightness at both minima we have another relation between k and a_0, of the form

$$a_0 = 1 - \lambda_1 + \frac{1-\lambda_2}{k^2}, \tag{7}$$

where, as always, λ_1 represents the light-intensity of the system at the middle of the eclipse of the smaller star by the larger.

These two equations must be solved for a_0 and k. As good a way as any is to compute a_0 from (7) for equidistant values of k, then take $\chi(k, a_0, \frac{1}{4})$ from Table III and find by interpolation what value of k satisfies (19). A graphical solution may be made by plotting on one sheet (a) the curves $\chi(k, a_0, \frac{1}{4}) = \text{const.}$, with k and a_0 as co-ordinates, and on another transparent sheet (b) the curves $a_0 = \frac{\text{const.}}{k^2}$. If these diagrams are superposed so that the point $a_0 = 0$, $k = 1$, on (b) lies above the point $a_0 = 1 - \lambda_1$, $k = 1$ on (a), then the intersection of the curves $\chi = \frac{C}{D}$ on (a) and $a_0 = \frac{1-\lambda_2}{k^2}$ on (b) will give the desired solution.

This graphical method has the advantage of exhibiting very clearly the degree of uncertainty of the results. When λ_1 and λ_2 are nearly equal the curves in question on (a) and (b) run very nearly parallel for a considerable part of their course (corresponding to the larger values of k), and the solution is usually very nearly indeterminate. If the two minima are considerably unequal, the two sets of curves cut at a considerable, though usually an acute, angle and the solution is determinate, and in most cases unique. When, however, $\lambda_2 > 0.60$ and $\lambda_1 > 0.70$, there may be two solutions, and if these lie near together, their determination becomes very uncertain. These indeterminate cases, it should be noticed, can arise only when the loss of light at minimum is less than half the original light, that is, when the minimum is less than $0^{\mathrm{m}}.75$ in depth.

When the principal minimum corresponds to the eclipse of the smaller star by the larger (that is, when $\lambda_1 < \lambda_2$) the curve (b) is steeper than the curve (a) (a_0 being the horizontal co-ordinate). If the two curves intersect for some value of k less than unity, the

curve (*b*) must give the greater value of a_0 when $k=1$. When $\lambda_2<\lambda_1$, the curve (*a*) is steeper than (*b*) and the reverse is the case. Now the equation of (*a*) is (19), and of (*b*) is (7). When $k=1$ the latter gives $a_0=2-\lambda_1-\lambda_2$. The value of a_0 which satisfies the former may be found in any given case from Table III. Call it β. Then if $2-\lambda_1-\lambda_2>\beta$ the smaller star is behind the larger at principal eclipse; and vice versa. If these two quantities are equal, $k=1$, that is, the two stars are of the same radius. When both minima are of very small depth (say $0^m.15$ or less), the indeterminate character of the solution is pronounced (except when k is near its lower limit, and the eclipse is nearly total). The exact form of the light-curve for so shallow a minimum is especially difficult to fix by observation. It follows that the eclipsing variables of small range, such as have been recently discovered with the selenium photometer, present a problem which can be solved only by the addition of other data than those derivable from the light-curve. Fortunately, it is in just these cases that spectroscopic data make it possible to estimate the ratio of the actual light-emissions of the components, and also determine with certainty which component is in front at each eclipse. The ratio of the surface-intensity of the components being that of the loss of light at the two eclipses, k may at once be determined.[1]

Whenever, by any of these means, k and a_0 have been found, the determination of the remaining elements is simple. At the middle of eclipse $\theta=0$, and the apparent distance of centers is $(1+pk)$ times the radius of the larger star, where p is the function of k and a_0, defined on page 320, whose values are given in Table I. The value of θ at the beginning or end of the eclipse may be computed by (18) setting $n=0$. Calling this θ', we have, as before,

$$r_1^2(1+k)^2=\cos^2 i+\sin^2 i \sin^2\theta',$$

which may be written

$$\left.\begin{aligned} r_1^2(1+k)^2&=\cos^2 i \cos^2\theta'+\sin^2\theta'. \\ &\text{For the middle of the eclipse, we have} \\ r_1^2(1+pk)^2&=\cos^2 i\,. \end{aligned}\right\} \quad (20)$$

[1] F. Schlesinger, *Publications of the Allegheny Observatory*, 2, 56.

These two equations determine r_1 and $\cos i$. The relations $r_2 = kr_1$, $a_0 L_2 = 1 - \lambda_1$, determine the remaining elements.

The above method of solution of the problem presented by a partial eclipse, though not rigorously exact, is sufficiently accurate to be adopted as final, unless the observations are unusually numerous and precise. If it seems desirable to proceed farther, the light-curve which corresponds rigorously to the assumed elements may be computed by the formulae

$$\sin^2 \theta = A + B\psi(k, a), \tag{13}$$

$$1 - l = \frac{a}{a_0}(1 - \lambda), \tag{21}$$

where A and B are derived from r_1, k, and i by means of the equations (14) (p. 323). The light-curves for both primary and secondary minima should be computed, using the appropriate values of λ in (21). If they do not represent the observations sufficiently well, the assumed constants may be varied—in which case it is well to take as fundamental data λ_1, λ_2 (the light-intensities at the middle of the two minima), k, and A, determining a_0 by the equation (7) and then B by the equation

$$A + B\psi(k, a_0) = 0. \tag{22}$$

As before, the elements need not be computed until a satisfactory light-curve has been found.

PART II. CONSTRUCTION OF THE TABLES

All the tables necessary for computing the elements of eclipsing variables may be derived from Table I. This gives a function of two variables, $p\ (k, a)$, which may be defined as follows.

Let a circle of radius unity cut a smaller circle of radius k in such a way that the area of the interior segment of the latter is a times that of the whole circle. The distance of the centers of the two circles is then $1 + kp(k, a)$; or p is the ratio which the distance of the center of the smaller circle from the circumference of the larger bears to the radius of the former. For any fixed value of k, a may be computed when p is known by familiar equations; but it saves much work to compute corresponding values of a and p for assigned values of some other variable. In the present

TABLE I

RELATION BETWEEN THE ECLIPSED AREA α AND THE DISTANCE OF CENTERS δ

α	k=1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
0.00	+1.000	+1.000	+1.000	+1.000	+1.000	+1.000	+1.000	+1.000	+1.000	+1.000	+1.000
.01	0.919	0.921	0.922	0.924	0.925	0.927	0.929	0.930	0.932	0.934	0.935
.02	.868	.871	.873	.876	.879	.881	.884	.887	.890	.892	.895
.05	.755	.759	.764	.769	.774	.779	.785	.790	.795	.800	.805
.10	.610	.618	.624	.631	.638	.645	.653	.661	.670	.678	.687
0.15	+0.488	+0.496	+0.504	+0.513	+0.523	+0.533	+0.544	+0.554	+0.565	+0.576	+0.585
.20	.374	.388	.398	.408	.419	.430	.443	.456	.469	.481	.492
.25	.267	.284	.297	.310	.322	.335	.348	.363	.378	.391	.405
.30	.168	.186	.200	.216	.230	.244	.258	.272	.288	.303	.321
.35	+ .075	.094	.110	.127	.143	.160	.175	.190	.207	.222	.239
0.40	−0.015	+0.005	+0.024	+0.041	+0.059	+0.077	+0.094	+0.109	+0.126	+0.143	+0.159
.45	− .106	− .081	− .061	− .042	− .023	− .004	+ .013	+ .028	+ .045	+ .062	+ .079
.50	− .194	− .166	− .145	− .124	− .103	− .084	− .067	− .051	− .034	− .017	− .000
.55	− .280	− .250	− .226	− .204	− .184	− .165	− .148	− .131	− .113	− .096	− .079
.60	− .364	− .332	− .306	− .284	− .263	− .244	− .226	− .209	− .192	− .175	− .159
0.65	−0.447	−0.413	−0.386	−0.363	−0.343	−0.323	−0.305	−0.288	−0.271	−0.255	−0.239
.70	− .528	− .492	− .465	− .441	− .420	− .401	− .383	− .367	− .350	− .336	− .321
.75	− .607	− .571	− .544	− .520	− .498	− .481	− .463	− .448	− .432	− .419	− .405
.80	− .686	− .649	− .622	− .600	− .580	− .563	− .546	− .532	− .517	− .504	− .492
.85	− .765	− .728	− .701	− .680	− .663	− .648	− .633	− .620	− .607	− .596	− .585
0.90	−0.843	−0.807	−0.783	−0.764	−0.749	−0.736	−0.725	−0.715	−0.705	−0.696	−0.687
.95	− .922	− .890	− .872	− .858	− .847	− .838	− .830	− .823	− .817	− .811	− .805
.98	− .967	− .945	− .935	− .928	− .922	− .915	− .910	− .905	− .900	− .896	− .892
.99	− .983	− .967	− .960	− .955	− .951	− .948	− .945	− .942	− .939	− .937	− .934
1.00	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000

The tabular quantity is $p(k, \alpha)$, $\delta = r_1(1+kp)$.

case the common chord of the two circles was chosen. Call this $2s$. We then have at once for the distance of centers $1+kp=\sqrt{1-s^2}\pm\sqrt{k^2-s^2}$, the upper or lower sign being employed according as the center of the smaller circle is outside or inside the chord. Let A_1 and A_2 be the areas of the smaller segments cut from the two circles by this chord. Then, when the center of the smaller circle is outside the chord, we have

$$a=\frac{A_1+A_2}{\pi k^2},$$

otherwise

$$a=1-\frac{A_2-A_1}{\pi k^2}.$$

If $s=\sin\phi_1=k\sin\phi_2$, we have

$$A_1=\phi_1-\sin\phi_1\cos\phi_1, \qquad A_2=k^2(\phi_2-\sin\phi_2\cos\phi_2).$$

The computation of these angles may be avoided by introducing the heights of the two segments, which are defined by the equations

$$h_1=1-\sqrt{1-s^2}=1-\cos\phi_1, \qquad h_2=k-\sqrt{k^2-s^2}=k(1-\cos\phi_2).$$

We then have

$$A_1=sh_1\frac{\phi_1-\sin\phi_1\cos\phi_1}{\sin\phi_1-\sin\phi_1\cos\phi_1}, \qquad \frac{h_1}{s}=\frac{1-\cos\phi_1}{\sin\phi_1}=\tan\tfrac{1}{2}\phi_1,$$

and similarly for A_2 and h_2.

If we set

$$\frac{\phi-\sin\phi\cos\phi}{\sin\phi-\sin\phi\cos\phi}=F(\tan\tfrac{1}{2}\phi),$$

the expressions previously found for a and p become

$$a=\frac{s}{k_2}\left\{h_1F\left(\frac{h_1}{s}\right)+h_2F\left(\frac{h_2}{s}\right)\right\}, \qquad p=1-\frac{h_1+h_2}{k};$$

$$a=1-\frac{s}{k_2}\left\{h_2F\left(\frac{h_2}{s}\right)-h_1F\left(\frac{h_1}{s}\right)\right\}, \qquad p=\frac{h_2-h_1}{k}-1.$$

The function F changes slowly, and a very small table suffices for it.

Every assumed value of s gives two points in the light-curve, that is, two pairs of values of a and p. By plotting these points on a suitable scale, the values of p corresponding to given values of a can be read off. A ten-inch slide-rule was used in the actual

TABLE II

FOR USE IN CASE OF TOTAL ECLIPSE. VALUES OF $\psi(k, \alpha_1)$

α_1	k=1.00	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10	0.00
0.00	+9.464	+7.478	+6.200	+5.279	+4.556	+3.984	+3.503	+3.104	+2.755	+2.454	+2.199
.02	8.095	6.457	5.373	4.606	4.000	3.504	3.106	2.768	2.478	2.216	2.000
.05	7.042	5.616	4.704	4.047	3.534	3.118	2.777	2.488	2.241	2.017	1.829
0.10	+5.759	+4.625	+3.895	+3.364	+2.960	+2.627	+2.358	+2.131	+1.934	+1.754	+1.603
.15	4.755	3.839	3.248	2.826	2.504	2.240	2.024	1.841	1.682	1.537	1.412
.20	3.906	3.184	2.712	2.374	2.110	1.898	1.726	1.581	1.453	1.336	1.235
0.25	+3.158	+2.600	+2.232	+1.969	+1.760	+1.591	+1.453	+1.344	+1.242	+1.146	+1.070
.30	2.522	2.088	1.803	1.603	1.443	1.314	1.205	1.115	1.039	0.968	0.911
.35	1.979	1.641	1.425	1.276	1.157	1.061	0.982	0.911	0.854	0.797	.756
0.40	+1.490	+1.245	+1.087	+0.978	+0.894	+0.825	+0.770	+0.721	+0.675	+0.633	+0.604
.45	1.040	0.881	0.777	.705	.649	.603	.566	.530	.501	.473	.453
.50	0.648	.555	.491	.451	.418	.392	.370	.348	.331	.314	.302
0.55	+0.300	+0.258	+0.233	+0.217	+0.202	+0.191	+0.181	+0.171	+0.164	+0.156	+0.151
.60	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
.65	− .258	−0.231	−0.214	−0.202	− .191	−0.181	−0.174	−0.167	−0.160	−0.156	−0.152
0.70	−0.480	−0.435	−0.408	−0.387	−0.369	−0.354	−0.344	−0.331	−0.320	−0.314	−0.306
.75	− .660	− .613	− .584	− .558	− .539	− .522	− .508	− .494	− .483	− .475	− .465
.80	− .805	− .765	− .738	− .717	− .700	− .684	− .670	− .659	− .647	− .639	− .632
0.85	−0.922	−0.893	−0.877	−0.863	−0.854	−0.843	−0.833	−0.825	−0.818	−0.812	−0.808
.90	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000
.95	−1.045	−1.085	−1.112	−1.134	−1.152	−1.166	−1.179	−1.190	−1.203	−1.214	−1.226
0.98	−1.0625	−1.126	−1.176	−1.220	−1.256	−1.284	−1.308	−1.329	−1.350	−1.369	−1.391
.99	−1.0643	−1.139	−1.199	−1.250	−1.293	−1.328	−1.362	−1.390	−1.419	−1.444	−1.471
1.00	−1.0650	−1.155	−1.231	−1.297	−1.354	−1.402	−1.445	−1.484	−1.525	−1.556	−1.596

work, and the computation of 21 points, sufficient to define the curve for a given value of k, took only about fifteen minutes. Table I as here printed has been carefully checked by differences, both horizontal and vertical, and the errors of the tabular quantities ought not to exceed one or two units of the last decimal place. The tabular interval has been made small enough to permit linear interpolation in both components except for very small or very large values of a.

Table II contains the function $\psi(k, a_1)$ defined by the equation

$$\psi(k, a_1)=\frac{\{1+kp(k, a_1)\}^2-\{1+kp(k, a_2)\}^2}{\{1+kp(k, a_2)\}^2-\{1+kp(k, a_3)\}^2},$$

(where $a_2=0.6$ and $a_3=0.9$), which is used in determining k in the case of total eclipse. The uncertainty of the tabular quantities does not exceed one or two units of the last decimal place, except for the larger values of ψ, corresponding to values of a_1 less than 0.3, for which the actual errors may be greater, but are not more serious in proportion to the whole quantity tabulated.

Table II*a* contains the functions

$$\phi_1(k)=\frac{4k}{\psi(k, 0)-\psi(k, 1)} \quad \text{and} \quad \phi_2(k)=\frac{4k}{(1-k)^2\psi(k, 0)-(1+k)^2\psi(k, 1)},$$

which are useful in determining the elements in the case of total eclipse.

Table III contains the function

$$\chi(k, a_0, \tfrac{1}{4})=\frac{\psi(k, \frac{1}{4}a_0)-\psi(k, a_0)}{\psi(k, \frac{1}{2}a_0)-\psi(k, a_0)},$$

which is of use in the case of partial eclipses. The accuracy of the tabular quantities is comparable with that in the previous tables. Table III*a* contains the functions $\omega_1(n)$ and $\omega_2(n)$ which appear as coefficients in the empirical relation

$$\chi(k, a_0, n)=\omega_1(n)+\omega_2(n)\chi(k, a_0, \tfrac{1}{4}),$$

which has been found to represent the individual computed values with such remarkable approximation. The values resulting from

TABLE II*a*

FOR COMPUTING THE ELEMENTS IN THE CASE OF TOTAL ECLIPSE

k	$\phi_1(k)$	$\phi_2(k)$
1.00	0.380	0.939
0.95	.401	.894
.90	.417	.848
0.85	0.427	0.802
.80	.431	.755
.75	.431	.709
0.70	0.427	0.663
.65	.419	.617
.60	.406	.572
0.55	0.390	0.527
.50	.371	.482
.45	.349	.436
0.40	0.323	0.390
.35	.294	.345
.30	.262	.298
0.25	0.226	0.250
.20	.187	.202
.15	.145	.153
0.10	0.100	0.103
.05	.052	.052
.00	.000	.000

TABLE III

FOR USE IN CASE OF PARTIAL ECLIPSE. VALUES OF $\chi(k, \alpha_0, \frac{1}{4})$

α_0	k=1.00	0.95	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.20	0.10	0.00
1.00...	2.462	2.316	2.194	2.010	1.870	1.756	1.667	1.596	1.541	1.491	1.441	1.407
0.98...	2.402	2.270	2.168	2.005	1.872	1.770	1.690	1.628	1.572	1.522	1.479	1.437
.95...	2.328	2.208	2.121	1.985	1.872	1.779	1.703	1.638	1.586	1.540	1.499	1.460
.90...	2.218	2.135	2.062	1.950	1.860	1.778	1.708	1.650	1.605	1.560	1.522	1.485
.85...	2.127	2.067	2.010	1.919	1.840	1.771	1.710	1.660	1.618	1.577	1.540	1.501
0.80...	2.054	2.010	1.968	1.890	1.823	1.763	1.711	1.665	1.626	1.589	1.552	1.510
.70...	1.956	1.926	1.896	1.842	1.792	1.749	1.711	1.675	1.640	1.606	1.570	1.532
.60...	1.887	1.860	1.838	1.799	1.764	1.734	1.707	1.677	1.648	1.620	1.586	1.551
.50...	1.828	1.805	1.789	1.762	1.740	1.719	1.699	1.675	1.650	1.626	1.597	1.568
.40...	1.772	1.761	1.751	1.732	1.717	1.700	1.685	1.669	1.650	1.629	1.607	1.584
0.30...	1.727	1.722	1.717	1.709	1.700	1.688	1.674	1.661	1.646	1.635	1.615	1.598
.20...	1.693	1.692	1.690	1.687	1.682	1.675	1.665	1.655	1.642	1.635	1.623	1.608
.10...	1.673	1.672	1.670	1.666	1.660	1.655	1.650	1.646	1.639	1.633	1.626	1.616
.00...	1.630	1.630	1.630	1.630	1.630	1.630	1.630	1.630	1.630	1.630	1.630	1.630

TABLE III*a*

FOR COMPUTING THE FORM OF THE LIGHT-CURVE IN CASE OF PARTIAL ECLIPSE

n	$\omega_1(n)$	$\omega_2(n)$
0.00	−3.94	+4.10
.10	−1.45	+2.21
.20	−0.399	+1.330
.25	.000	+1.000
0.30	+0.316	+0.720
.35	+ .567	+ .488
.40	+ .758	+ .295
.45	+ .899	+ .133
0.50	+1.000	0.000
.55	+1.065	−0.107
.60	+1.095	− .190
.65	+1.090	− .249
0.70	+1.046	−0.285
.75	+0.967	− .297
.80	+ .846	− .285
.85	+ .693	− .250
0.90	+0.503	−0.191
.95	+ .273	− .108
.98	+ .114	− .047
.99	+ .058	− .024
1.00	0.000	0.000

the individual computations for different values of n have been carefully smoothed and the table is probably as trustworthy as the others. For $n=0$ and $n=\frac{3}{4}$, the values of $\chi(k, a_0, n)$ have been computed over the whole range of values of k and a_0; for the other values of n only for $a_0=0.80$ (which appeared in the previous computations to give results very closely approximating to the mean for all values of a_0).

Two auxiliary tables are added to facilitate numerical computation. Table A gives the *loss* of light $(1-\lambda)$, corresponding to a given change Δm in stellar magnitude.[1] For a difference of magnitude greater than 2.5, the loss of light is $0.9000+\frac{1}{10}$ of the tabular value for $\Delta m - 2^{\mathrm{m}}.5$. Table B gives the values of $\theta - \sin\theta$ for every 0.01 of θ (expressed in circular measure), and saves much labor

[1] S. Blazko, *op. cit.*, p. 106.

TABLE A

LOSS OF LIGHT CORRESPONDING TO AN INCREASE Δm IN STELLAR MAGNITUDE

Δm	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0092	0.0183	0.0273	0.0362	0.0450	0.0538	0.0624	0.0710	0.0795
.1	.0880	.0964	.1046	.1128	.1210	.1290	.1370	.1449	.1528	.1605
.2	.1682	.1759	.1834	.1909	.1983	.2057	.2130	.2202	.2273	.2344
.3	.2414	.2484	.2553	.2621	.2689	.2756	.2822	.2888	.2953	.3018
.4	.3082	.3145	.3208	.3270	.3332	.3393	.3454	.3514	.3573	.3632
0.5	0.3690	0.3748	0.3806	0.3862	0.3919	0.3974	0.4030	0.4084	0.4139	0.4192
.6	.4246	.4298	.4351	.4402	.4454	.4505	.4555	.4605	.4654	.4703
.7	.4752	.4848	.4848	.4895	.4942	.4988	.5034	.5080	.5125	.5169
.8	.5214	.5258	.5301	.5344	.5387	.5429	.5471	.5513	.5554	.5594
.9	.5635	.5675	.5715	.5754	.5793	.5831	.5870	.5907	.5945	.5982
1.0	0.6019	0.6055	0.6092	0.6127	0.6163	0.6198	0.6233	0.6267	0.6302	0.6336
1.1	.6369	.6403	.6435	.6468	.6501	.6533	.6564	.6596	.6627	.6658
1.2	.6689	.6719	.6749	.6779	.6808	.6838	.6867	.6895	.6924	.6952
1.3	.6980	.7008	.7035	.7062	.7089	.7116	.7142	.7169	.7195	.7220
1.4	.7246	.7271	.7296	.7321	.7345	.7370	.7394	.7418	.7441	.7465
1.5	0.7488	0.7511	0.7534	0.7557	0.7579	0.7601	0.7623	0.7645	0.7667	0.7688
1.6	.7709	.7730	.7751	.7772	.7792	.7812	.7832	.7852	.7872	.7891
1.7	.7911	.7930	.7949	.7968	.7986	.8005	.8023	.8041	.8059	.8077
1.8	.8095	.8112	.8129	.8146	.8163	.8180	.8197	.8214	.8230	.8246
1.9	.8262	.8278	.8294	.8310	.8325	.8340	.8356	.8371	.8386	.8400
2.0	0.8415	0.8430	0.8444	0.8458	0.8472	0.8486	0.8500	0.8514	0.8528	0.8541
2.1	.8555	.8568	.8581	.8594	.8607	.8620	.8632	.8645	.8657	.8670
2.2	.8682	.8694	.8706	.8718	.8729	.8741	.8753	.8764	.8775	.8787
2.3	.8798	.8809	.8820	.8831	.8841	.8852	.8862	.8873	.8883	.8893
2.4	.8904	.8914	.8924	.8933	.8943	.8953	.8962	.8972	.8981	.8991
2.5	0.9000	0.9009	0.9018	0.9027	0.9036	0.9045	0.9054	0.9062	0.9071	0.9080

For values of Δm greater than 2.5, the loss of light is 0.9000 plus $\frac{1}{10}$ of the loss of light corresponding to $\Delta m-2.5$.

TABLE B

VALUES OF $\theta-\sin\theta$

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.00	0.0000	0.0002	0.0013	0.0045	0.0105	0.0206	0.0354	0.0558	0.0826	0.1167
.01	.0000	.0002	.0015	.0049	.0114	.0218	.0372	.0582	.0857	.1205
.02	.0000	.0003	.0018	.0055	.0122	.0231	.0390	.0607	.0888	.1243
.03	.0000	.0004	.0020	.0060	.0131	.0244	.0409	.0632	.0920	.1283
.04	.0000	.0005	.0023	.0066	.0141	.0258	.0428	.0658	.0953	.1324
0.05	0.0000	0.0006	0.0026	0.0071	0.0151	0.0273	0.0448	0.0684	0.0987	0.1365
.06	.0000	.0007	.0029	.0078	.0160	.0288	.0469	.0711	.1022	.1407
.07	.0001	.0008	.0033	.0084	.0171	.0304	.0490	.0739	.1057	.1450
.08	.0001	.0010	.0037	.0091	.0183	.0320	.0512	.0767	.1093	.1494
.09	.0001	.0011	.0041	.0098	.0194	.0337	.0535	.0796	.1130	.1539

in computing the values of sin θ corresponding to a given interval from minimum.

Examples of the use of these tables are found in a subsequent paper, in which elements are deduced for the eclipsing variables *W Delphini*, *W Ursae Majoris*, and *W Crucis*.

PRINCETON UNIVERSITY OBSERVATORY
March 19, 1912

[*To be continued*]

PLATE XVIII

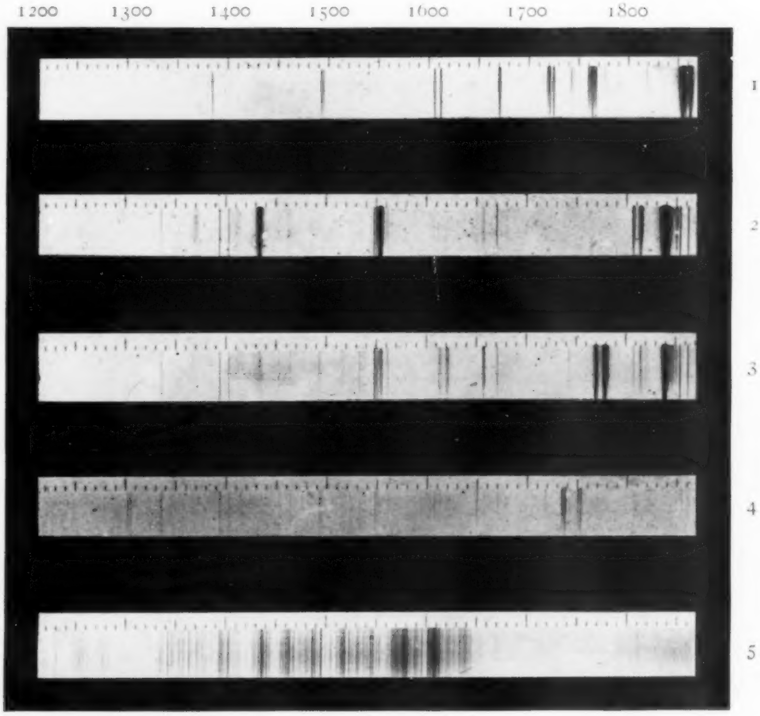

FIG. 1.—Aluminum
FIG. 2.—Calcium
FIG. 3.—Strontium
FIG. 4.—Magnesium
FIG. 5.—Hydrogen vacuum tube

SPARK SPECTRA OF THE ALKALI EARTHS IN THE SCHUMANN REGION

By THEODORE LYMAN

Spark spectra of metals in the region of extremely short wave-lengths have been investigated by Schumann and later by Handke.[1] The substances studied by the second observer included aluminum, copper, gold, silver, tin, zinc, magnesium, and mercury. The spark was in air outside the instrument. Schumann's vacuum prism spectroscope was used. The measurements extended to the neighborhood of λ 1600. The writer has made experiments on metallic spectra from time to time during the past ten years, using his grating vacuum spectroscope, but without important results until recently, when his attention was directed to the alkali earths by the work of Saunders on series spectra.[2]

The chief improvements which distinguish the present work consist, first, in the employment of a concave grating in place of a prism, and second, in running the spark in a vessel through which a current of hydrogen was maintained. The light thus produced passed directly from the spark chamber through a fluorite window into the body of the spectroscope. In this way, the absorption of the layer of air between the spark and the window was eliminated. At the same time the current of hydrogen helped to free the light-path from the gaseous products of the spark.

Aluminum (Plate XVIII, Fig. 1) was chosen as the first substance for investigation because its spectrum has been studied by Handke and it therefore afforded opportunity for comparing the writer's measurements with measurements obtained by another type of instrument. The figures which will be found in Table I at the end of this paper illustrate in some degree the relative advantages of the two methods of experiment. The greater light-intensity of the prism instrument in the region of wave-lengths less refrangible than λ 1600 is shown by the fact that some faint lines are given by

[1] Inaug. Dis. Berlin, 1909.

[2] *Astrophysical Journal*, **32**, 153, 1910.

Handke which are not easily observed with the grating. On the other hand, the result of the elimination of the absorption of the fluorite prism and lenses and of the air near the spark is illustrated by the fact that the lines of shorter wave-length than λ 1600 were not discovered in the earlier work. Between the two sets of measurements, the agreement is fair; with a few exceptions, the difference between Handke's values and those of the writer are four-tenths of an Ångström unit.

Turning to the principal subject of this paper, the spectra of the alkali earths and their series relations, the expectations in the Schumann region based on theoretical considerations are of two kinds.[1] First, according to the speculations of Ritz and of Saunders there should be series of pairs in the region of very short wave-lengths, the subordinate members of which show constant wave-number separations. Lines belonging to this arrangement have already been observed over the ordinary extent of the spectrum in calcium, strontium, barium, and magnesium, and in the case of some of the series the constants of the formulae have been calculated with a sufficient degree of accuracy to permit of a rough prediction of the position as well as the separation of the new pairs.

The expectation of the second kind is less definite in character than the first. It is founded on the following statement of Saunders:[2] "In all three elements there occurs a strong pair in the ultra-violet: *Ba* λ 2335 and λ 2304; *Sr* λ 2165 and λ 2152; *Ca* λ 1840 and λ 1837, which are reversed in *Sr* and *Ba*, and probably in *Ca* also, and the line of greater wave-length is the stronger in each. They therefore look like subordinate-series pairs in a series of great strength, the rest of which is in the Schumann region."

Confining the attention at present to calcium (Plate XVIII, Fig. 2), the writer's experiments reveal four new pairs, with the separation required by theory, wave-number $1/\lambda = 223$. They appear to fall in with the expectation of the first kind and form terms in the first and second subordinate series. Their discovery is of considerable interest. In the case of the expectation of the second kind, the writer has found three new pairs with the

[1] Ritz, *Physikalische Zeitschrift*, **9**, 521, 1908.

[2] *Astrophysical Journal*, **32**, 165, 1910.

separation predicted by Saunders, $1/\lambda=70$. These pairs, four in all, may present material for the application of the "combination principle." After lines due to impurities have been eliminated, there remain some ten or twelve lines in calcium unclassified, most of which are faint.

In the case of strontium (Plate XVIII, Fig. 3), the expectation of the first kind is slight. The computations of Saunders show that the limits of the subordinate series lie in the neighborhood of λ 1700; it is a recognized property of these series that the members rapidly decrease in intensity as they near the limit. It is not surprising, therefore, that the writer has observed only one pair with the required separation, $1/\lambda=800$, in the Schumann region. Their position is λ 1847 and λ 1820. The expectation of the second class, however, has been well fulfilled. There are two striking pairs, with the separation $1/\lambda=285$.

These three pairs constitute the spectrum of strontium; the other lines visible on the plate are chiefly due to calcium and aluminum.

With barium the experimental difficulties are very great. Success was attained only by using the pure metal in an atmosphere of helium. What has been said of strontium applies even more strongly in this case. The expectation of the first kind is very small. The expectation of the second class is apparently fulfilled: two pairs with the separation predicted by Saunders exist. That they belong together is not absolutely certain, though it seems extremely probable. In addition to these lines, there are several others which may be due to barium.

Finally, the spectrum of magnesium has been studied (Plate XVIII, Fig. 4). It consists, in the Schumann region, of only two pairs, separation $1/\lambda=90$; they seem to fulfil the expectations of the first class and form members of the first and second subordinate series mentioned by Ritz.[1] The other lines visible on the plate are due to impurities.

In Plate XVIII, the fifth figure shows the vacuum tube spectrum of hydrogen which is added for the sake of comparison. Unfortunately the plate on which the barium spectrum is recorded is not suitable for reproduction.

[1] *Physikalische Zeitschrift*, **9**, 528, 1908.

It is hoped that the results already obtained are of interest in themselves, and it must be remembered that the method may be extended to the study of metallic spectra in general in the region bounded by λ 1850 at the one end and by the absorption of fluorite at the other.

The writer has profited by the advice of Professor Saunders throughout the work. In fact, if it had not been for Professor Saunders' interest in the subject of series spectra in the Schumann region, the research would never have been undertaken.

The details of technique and measurement are to be found in the second part of the paper.

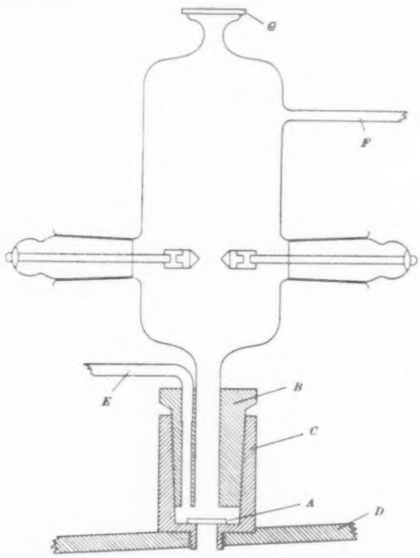

FIG. 1.—The spark chamber

The shape of the spark chamber is illustrated in the accompanying figure (see Fig. 1). A cylindrical glass vessel about 8 cm long and 4 cm in diameter is furnished at one end with a window, and is drawn out into a tube 1 cm in diameter, at the other. This tube fits into a brass cone *B* and is held in place by cement. The cone, in turn, fits air-tight into a cup *C* of the same form as that used in previous work.[1] The cup screws on to the face plate of the vacuum

[1] *Astrophysical Journal*, **32**, 102, 1911.

spectroscope D and has attached to it a fluorite window A. The point of novelty consists of a tube E which penetrates the side of the cone and through which a stream of hydrogen is directed into the spark chamber. This stream of gas, after striking the fluorite window, turns and is directed against the spark discharge and finally makes its way through sulphuric acid into the outer air by means of F. By a suitable arrangement of stop-cocks, the spark chamber may be exhausted through E. When it was found necessary to replace the stream of hydrogen by an atmosphere of helium, the gas was introduced through the tube F after the vessel had been thoroughly exhausted.

In exhausting the spark chamber and the body of the spectroscope itself, the excellent pump made by the Tri-Mount Rotary Power Company of Boston, has recently been employed.

The metal under examination is held in suitable clamps and introduced into the chamber through ground joints as shown in the figure. The discharge is viewed through the window G during the course of the experiment. Of late, a Hilger wave-length prism spectrometer has been used for the purpose.

The spark was produced by the Clapp Eastham transformer mentioned elsewhere.[1] In all cases, a capacity of 0.026 microfarads was used in parallel with the coil. In many cases an external spark gap was employed in series with the terminals in the discharge chamber.

It is obvious from what is known of the transparency of gases in the Schumann region that, as the light must pass over a path of more than 6.5 cm between spark and window, the gas in the discharge chamber must be free from impurities, especially oxygen. Hydrogen made from zinc and hydrochloric acid and by electrolysis from a barium hydroxide solution was employed. In both cases the usual precautions as to purification and drying were taken. The gas obtained by the second method gave the better results.

The helium was obtained from Tyrer of London and was used at a pressure of three quarters of an atmosphere. It exhibited a transparency equal if not superior to that of hydrogen. The writer believes that this is the first time the transparency of hydrogen in

[1] *Ibid.*

the Schumann region at atmospheric pressure in a column over 6 cm long has been demonstrated spectroscopically.

The method of measurement was that of shifted spectra, which has already been described in connection with the writer's work.[1] Briefly, it consists in comparing the spectrum under investigation with a known spectrum, generally of iron, which is "shifted" with respect to the first by a given amount. Lately, the process of measurement has been facilitated by the use of a scale specially etched on glass and standardized by comparison with the lines of the hydrogen spectrum. As it was an object to obtain as strong a spectrum as possible, a rather wide slit was used. The definition, therefore, falls short of that obtained in the author's hydrogen plates and in some of his gas spectra work.

The errors are of two kinds: one, which depends on the nature of the method employed, is connected with uncertainties in determining the magnitude of the "shift." This error affects the absolute values of the wave-lengths. With a sharp line in the spectrum of hydrogen it should not exceed 0.2 unit. The other is purely an error of setting and affects the relative values, such as the distance between the pairs. Its magnitude varies naturally with the nature of the line, its minimum is probably about 0.1 unit, its maximum may reach 0.4 in the case of nebulous lines. It is obviously this second error which must be borne in mind in estimating the accuracy of differences in the vibration numbers of the pairs of the same series. It is hardly necessary to add that measurements in the extreme ultra-violet are not well adapted for very accurately determining the constants of the formulae employed in series spectra work.

Impurities in the material employed in the study of spark spectra in the Schumann region may play a double rôle: they may contribute lines to the spectrum and they may form gaseous compounds in the spark chamber which modify the spectrum by their absorption. The metals used were analyzed in the hope that the data thus obtained might afford a clue to the cause of the anomalies observed during the experiment. However, the spectroscope is so delicate a detector of impurities that the results of chemical analysis

[1] *Astrophysical Journal*, **23**, 202, 1906.

give only a rough indication of what may be expected. A detailed discussion of the spectra follows.

The specimens of aluminum employed were of commercial quality but of considerable purity. Judging by past experience, they might be supposed to contain only traces of silicon, carbon, and iron. The spectrum in the visible region was dominated by the four lines of hydrogen; the lines of aluminum were relatively feeble. When the spark was tried in helium, the lines of that gas also dominated those of the metal. It is important to notice, however, that in all the spark spectra in the Schumann region, there are no lines which can be traced with certainty either to hydrogen or to helium.

Fig. 1 illustrates the spectrum of aluminum in hydrogen. All the strong lines on the plate are to be ascribed to the metal with the exception of the two pairs at λ 1742.7, 1745.3, and λ 1494.8, λ 1492.8, which are due to a trace of nitrogen.[1] Of the fainter lines, hardly visible in the plate, the one near λ 1655 is probably due to carbon, those near λ 1550 are found in magnesium but may be due to iron, those between λ 1490 and λ 1400 are common to several substances.

Turning to the tables, it is to be noted, that, for most of the lines, the differences between the values obtained by Handke and the measurements of the writer lie between three and four Ångström units. There is an uncertainty in determining the "shift" which introduces the possibility of an error into the writer's values amounting, at most, to two-tenths of an Ångström unit. This error, if it exists, would be of such a sign as to bring the writer's values into closer agreement with those of Handke. However, in the case of the nitrogen pairs at λ 1742.7, λ 1745.3, and in the case of the three lines λ 1721.2, λ 1719.3, and λ 1718.3, the difference in values rises to nearly a whole unit; this fact points to errors of measurements in the work of the earlier observer which may be present in all his data. In the case of the faint lines between λ 1854 and λ 1819, Handke's values alone are given, for although these lines are found on the writer's plate, their rather nebulous character renders their exact measurement difficult.

[1] *Astrophysical Journal*, **33**, 98, 1911.

Very recently the writer has made a new test with the aluminum spark in air. The experiment yielded more interesting results than any of the many previous attempts of this character, its success being due to the fact that the pointed spark terminals were placed nearly in contact with the window of the spectroscope, so that the spark played against the surface of the fluorite. An exposure of six minutes destroyed the fluorite window, but a strong spectrum was registered on the photographic plate. Between λ 1400 and λ 1900 the lines of this spectrum are the same as those observed when the spark was in hydrogen, but there is a distinct difference in the distribution of intensities in the two spectra in this region, owing to the selective nature of the absorption of the air. Near λ 1300, with the spark in air, there is a group of strong lines not observed when the spark was in hydrogen. Of these lines, those at λ 1302.0, λ 1304.8, λ 1305.8, λ 1334.6, and λ 1335.7 are obviously due to some common impurity for they are found with magnesium in hydrogen and with aluminum in helium. The other lines of the group are included in the Aluminum Table but are marked (*) to indicate the uncertainty of their origin.

The specimen of calcium was analyzed with the following result: iron 0.07 per cent, aluminum, magnesium, and silicon, a trace, no trace of barium or strontium or of any other common metal. The spark behaves well in an atmosphere of hydrogen. In Fig. 2 the most striking objects are the strong series of narrow pairs predicted by Saunders. They appear in the illustration as the four broad lines beginning near λ 1370 and ending near λ 1840. On the plate itself, their separation can be measured but their character renders the setting error large, especially in the case of the most refrangible member.

Of the four pairs with wave-number separation $1/\lambda=223$, predicted by Ritz, only the two at the less refrangible end of the plate are visible in the illustration and of these one member of the first pair is concealed by the strong narrow pair at λ 1840. The other two pairs belonging to this series, though weak, are clearly seen on the original plate. Of the other lines, the sharp narrow pair near λ 1870 may belong to calcium, the lines at λ 1854 and λ 1862 are evidently due to aluminum, the line near λ 1670

belongs to aluminum but seems unduly enhanced here. The nebulous narrow pair at λ 1657 is visible in most metallic spectra both in helium and hydrogen; it is due to a common impurity—silicon or carbon, perhaps. The very faint line near λ 1650 also does not belong to calcium. The background of lines, hardly visible in the illustration, between this point and λ 1555.1 is found in iron, as is also the sharp narrow pair near λ 1550 next to the strong calcium group. The characteristic sharp pair at λ 1393, λ 1402 may be due to calcium but appears in other metals more strongly than their calcium content would seem to warrant. The narrow pair near λ 1335 appear in several metals both in helium and hydrogen. The extreme group not visible in the illustration is found only with calcium; it is made up of some of the most refrangible lines yet obtained with a metal.

Some difficulty was found in obtaining metallic strontium of sufficient purity. A suitable specimen was finally procured, however. It gave the following analysis:

Silicon	None	Calcium	1.5 per cent
Iron	Slight trace	Magnesium	None
Aluminum	Trace	Mercury	None
Barium	None	Carbon	Considerable, not estimated

On inspection of the spectrum, it is obvious that most of the lines are also found in calcium. The only strong lines that can be attributed to strontium are the two pairs at λ 1769, λ 1778, λ 1613, and λ 1620. They also appeared faintly in a spectrum obtained with a less pure specimen of metal.

There is also a weak pair near λ 1821, λ 1847, one member of which may be visible in the illustration, which is predicted by Ritz and Saunders; it has the correct wave-number separation $1/\lambda = 800$.

The magnesium employed yielded the following analysis:

MAGNESIUM

Silicon	None	Barium	None
Iron	0.05 per cent (estimated colorometrically)	Strontium	None
Aluminum	Trace	Calcium	None

On comparing its spectrum with one obtained with iron terminals under similar conditions, it appears that only the two pairs already mentioned can be attributed to magnesium. They appear near λ 1735 and λ 1750. All the other lines are found in the iron spectrum and are due, therefore, either to it or to some impurity common to the two substances. In this connection, it is interesting to observe how small an amount of an impurity may produce an appreciable effect.

The pairs that might be expected at the more refrangible end of the plate cannot be distinguished with certainty from the background of faint lines.

The results of Handke's investigation of the magnesium spectrum do not agree with those of the writer. Perhaps the difference in the condition of the spark in the two cases may account for this.

In the case of barium, the most obvious mode of procedure is to employ carbon terminals saturated with some salt of the metal. An experiment of this kind was therefore tried; carbon terminals saturated with a solution of chloride were used in an atmosphere of hydrogen. The spark showed the barium spectrum in the visible quite strongly, but nothing in the Schumann region.

Through the kindness of Professor T. W. Richards, the writer obtained some unusually pure specimens of barium, containing over 99 per cent of the metal. The impurities were a trace of oxygen and of iron. Experiments with this substance, however, in an atmosphere of hydrogen gave no result at all, perhaps because of the formation of an absorbing cloud of hydride about the spark. Even the results obtained with the spark in helium were somewhat disappointing. The members of the pair λ 1849, λ 1869 have the correct separation $1/\lambda=575$ and relative intensity, but they are very feeble and rather sharper than one would be led to expect from similar pairs in calcium and strontium. Another pair with correct separation and of a more satisfactory appearance can be picked out from the remaining lines at λ 1678, λ 1694.

The spectrum of hydrogen, which is added for the sake of comparison, was obtained from a vacuum tube in the manner already described.[1] There was no capacity other than that of the leads

[1] *Astrophysical Journal*, **23**, 202, 1906.

in the circuit. The pressure of the gas was about two millimeters. The faint lines on the less refrangible side of λ 1650 are due to a trace of an oxide of carbon.

The similarity between the hydrogen and magnesium spectrum in the neighborhood of λ 1600 is apparent, not real. The two sharp lines, which have been mentioned as occurring most strongly in calcium and whose wave-lengths are λ 1393.6, λ 1402.7, nearly agree in position with the two lines in hydrogen λ 1394, λ 1402.8. It is barely possible, therefore, that these lines are due to the gas, though, as they occur faintly with aluminum in helium, it does not seem probable. If they belong to hydrogen, they are the only lines in the vacuum tube spectrum which appear in the spark.

As usual, the wave-lengths on the table are in vacuum. The scale which is printed with the spectra was not used directly for measurements. It is intended to give the position of the lines only approximately. Of the lines obviously due to impurities the most prominent are given in Table VI.

TABLE I

ALUMINUM

λ	I	1/λ	λ Handke	Diff.	λ	I	1/λ	λ Handke	Diff.
1238.8*...	1	80723	......	....	1750.0...	3	57143	1750.4	0.4
1264.5*...	1	79083	......	....	1751.7...	2	57087	1752.1	.4
1275.0*...	3	78431	......	....	1760.0...	8	56818	1760.4	.4
1276.4*...	2	78345	......	....	1761.9...	8	56757	1762.4	.5
1310.8*...	6	76290	......	....	1763.8...	10	56695	1764.2	.4
1319.4*...	6	75792	......	....	1765.7...	8	56635	1766.0	.3
1326.6*...	1	75380	......	....	....	..	.....	1766.9	..
1343.4*...	2	74438	......	....	1767.6...	8	56574	1768.0	.4
1352.8...	1	73921	......	....	....	..	.....	1769.6	..
1379.5...	3	72490	......	....	....	..	.....	1772.9	..
1383.9...	5	72259	......	....	1773.8...	2	56376	1773.8	.0
1540.1...	1	64931	......	....	....	..	.....	1774.9	..
1605.6...	8	62282	1605.9	0.3	1776.9...	4	56278	1777.1	.2
1611.8...	8	62042	1612.1	.3	....	..	.....	1777.8	..
1670.6...	10	59859	1671.0	.4	....	..	.....	1792.1	..
....	..	.....	1676.1	..	1818.5...	3	54990	1819.0	.5
1718.3...	1	58197	1719.1	.8	....	..	.....	1819.6	..
1719.3...	9	58163	1720.0	.7	....	..	.....	1820.6	..
1721.2...	9	58099	1722.0	.8	....	..	.....	1833.2	..
1725.0...	10	57971	1725.3	.3	....	..	.....	1836.8	..
....	..	.....	1741.1	..	1854.7...	50	.....	1854.7	..
1742.7	N.	57382	1743.6	.9	1858.2...	10	.....	1858.2	..
1745.3		57297	1746.3	1.0	1862.8...	50	.....	1862.8	..
1747.7...	1	57218	1748.3	.6					

TABLE II

CALCIUM

λ	I	$1/\lambda$	$\Delta 1/\lambda$	λ	I	$1/\lambda$	$\Delta 1/\lambda$
1246.2	1	80244	...	1553.5	7	64370	66
1254.3	2	79726	...	1555.1	8	64304	
1260.2	1	79352	...	1561.2	2?	64053	
1264.5	2	79083	...	1674.1	1	59733	227
1268.2	2	78852	...	1680.5	2	59506	
1276.4	3	78345	...	1692.4	1	59087	226
1369.1	3	73040	80	1698.9	2	58861	
1370.6	3	72960		1807.8	7	55316	220
1393.6	5	71761	...	1815.0	8	55096	
1402.7	4	71291	...	1838.0	9	54406	65
1433.1	5	69778	58	1840.2	10	54341	
1434.3	6	69720		1843.8	6	54236	220
1526.7	2	65501	...	1851.3	7	54016	
1533.4	2	65214	...	1870.4	3	53464	...
1546.0	?	64683	...	1872.5	3	53404	...

TABLE III

STRONTIUM

λ	I	$1/\lambda$	$\Delta 1/\lambda$	λ	I	$1/\lambda$	$\Delta 1/\lambda$
1532.3?	1	65261	...	1769.8	8	56503	286
1537.9?	1	65024	...	1778.8	9	56217	
1560.8?	1	64070	...	1820.0	1	54945	803
1613.3	4	61985	283	1847.0	3	54142	
1620.7	5	61702					

TABLE IV

BARIUM

λ	I	$1/\lambda$	$\Delta 1/\lambda$	λ	I	$1/\lambda$	$\Delta 1/\lambda$
1331.1	2	75126	...	1572.9	2	63577	...
1361.0	2	73475	...	1592.9	1	62778	...
1414.8	3	70681	...	1674.5	4	59719	...
1417.1	2	70567	...	1677.9	3	59598	577
1482.0	1	67476	...	1694.3	6	59021	
1485.0	1	67340	...	1786.6	1	55972	...
1487.0	2	67249	...	1849.5	2	54068	569
1503.9	4	66494	...	1869.2	5	53499	
1554.5	3	64329	...				

TABLE V

MAGNESIUM

λ	I	$1/\lambda$	$\Delta 1/\lambda$	λ	I	$1/\lambda$	$\Delta 1/\lambda$
1735.0	6	57637	93	1750.9	5	57113	88
1737.8	7	57544		1753.6	6	57025	
				1828.1	1	54702	..

TABLE VI

LINES OF UNCERTAIN ORIGIN

λ	I	1/λ	Δ1/λ	λ	I	1/λ	Δ1/λ
1302.0........	4	76805	..	1548.2........	6	64591	..
1304.8........	3	76640	..	1550.8........	5	64483	..
1305.8........	1	76581	..	1649.9........	3	60609	..
1334.6........	2	74929	..	1656.8........	4	60357	..
1335.7........	3	74867	..	1657.8........	1	60321	..

In conclusion it may be well to restate the results which have been achieved.

The existence of certain lines in that part of the spectra of the alkali earths which lie in the Schumann region was predicted by Ritz and Saunders on theoretical grounds. When these predictions were made, the spectra of the substances in the region of extremely short wave-lengths had never been observed. The writer has succeeded in photographing these spectra and he has discovered part, at least, of the lines whose existence was predicted. During the work, a method has been developed by which spark and arc spectra may be studied down to the very limit of the transparency of fluorite.

JEFFERSON PHYSICAL LABORATORY
HARVARD UNIVERSITY
May 1912

REVIEWS

Lines in the Arc Spectra of Elements. Compiled by F. STANLEY. London: Adam Hilger, Ltd., 1911. 8vo, pp. 140. Cloth, 12*s.* 6*d.*; half morocco, 15*s.* 6*d.*

This volume lists the wave-lengths and intensities of 3700 selected lines from the arc spectra of 55 elements arranged according to their wave-lengths. In adjoining columns are given the element and the wave-length of the next prominent line of that element. The printed matter occupies about one-half of a page, the remainder of the page and the opposite page being left blank for the addition of notes. The more persistent lines are denoted in many cases by an asterisk.

As the list is not exhaustive as to either element or line, and as the wave-lengths are rounded off to tenths of an Ångström and no references are given, it is not likely that the book will be of great value to the advanced worker in physics and astronomy, although it is probably well adapted to the less exacting needs of the chemist and of the amateur.

STORRS B. BARRETT

Physical Optics. By ROBERT W. WOOD. New and Revised Edition. New York: MacMillan, 1911. Pp. xvi+705. $5.25 net.

The first edition of Wood's *Physical Optics,* which appeared in 1905, was universally recognized as an important addition to scientific literature. The second edition is of distinctly greater value than the first. The book has been enlarged in size from 546 pages to 705 pages, and the number of figures has been increased from 325 to 399. The new colored frontispiece contains 8 figures as against 5 in the old edition, and there are 10 full-page plates in the new edition, twice the former number.

The most noticeable additions are contained in three short new chapters. Chap. xii, on meteorological optics, deals with the rainbow, halos, mock suns, and related phenomena. Chap. xix, on electro-optics, deals with the Kerr electro-optic effect in liquids, the electro-optic analogy of the Zeeman effect, and the photo-electric effect. In chap. xxv the author has attempted the difficult task of presenting the

principle of relativity in less than 11 pages. It is doubtful if a better treatment could be given in so small a compass.

The new material incorporated in the old chapters is, in part, as follows: To chap. i has been added a description of Pfund's mercury arc, a description of Galitzin and Willip's repetition of Belopolsky's experiment on the Doppler effect, Stark's work on the Doppler effect in the light emitted by the canal rays in vacuum tubes, and a figure showing the Doppler effect in stellar spectra.

The author's work on "fish-eye views" and Schmidt's theory of the sun are the chief additions to the fourth chapter. Julius' work on the effect of anomalous dispersion on the appearance of the D lines, illustrated by a full-page plate, is an interesting addition to chap. v. To chap. vii has been added a discussion of the author's echelette grating, and the work of Wood and Trowbridge on spectral intensity and the form of grooves. The interesting and valuable work of Rubens and Wood on the focal isolation of long heat-waves has been added to the chapter on the theory of dispersion. Chap. xv, on the absorption of light, has been expanded by the addition of Wood's work on the absorption of sodium vapor, and the extension of the Balmer series in the ultra-violet to the forty-eighth member. There is also reference to the very interesting work of Pflüger and of Ladenburg and Loria on absorption by luminous hydrogen. The chapter on magneto-optics has been increased by 18 pages and contains much new and interesting material, e.g., the work of Voigt and Lohmann on complicated types of the Zeeman effect, the work of Zeeman on unsymmetrical triplets, a discussion of the Zeeman effect in spectral series, the work of Zeeman and Winawer on the Zeeman effect in absorption spectra, Hale's discovery of the Zeeman effect in sun-spot spectra, and Dufour's recent work on the Zeeman effect in band spectra.

The chief additions to chap. xx are on Wood's investigations of the fluorescence of mercury, iodine, and bromine vapors.

These illustrations serve to show that most of the material added to the book is in the presentation of results obtained in physical optics since the first edition appeared.

As a whole the book, like its predecessor, deals primarily with experimental optics, and illustrates the practicability and perhaps also the desirability of treating even the most abstruse topics from a physical rather than a mathematical standpoint. And yet the mathematical treatment has not been ignored, and we find all of it that is essential to a profound knowledge of theoretical optics.

One hesitates to criticize adversely in any way a book of such unquestioned merit, but there are evidences of carelessness in editing which cannot be overlooked. We read in the preface to the second edition, "The numerous typographical errors which marred the first edition have been corrected and certain sections of small interest or importance have been removed bodily to make room for new material." It is, therefore, something of a disappointment to find that there are still a great number of typographical errors. To make matters worse, many of these errors were noted in the page of errata which accompanied the first edition. Thus on p. 383 the equation of line 2 is entirely wrong, the brackets have been omitted for line 5, the incorrect equation of lines 5, 6, and 7 from the bottom is reproduced from the first edition, the parenthesis is misplaced in line 4 from the bottom, and a minus sign is omitted between the last two lines.

On p. 1 the author refers to the last chapter of the book instead of to the next to the last chapter. On p. 10 he refers to p. 351, instead of p. 439, and on p. 35 to p. 158 instead of p. 191.

These references are to the page of the old edition instead of to the page of the new edition. On p. 15 the author refers to the Carnegie Institution of Washington as the Carnegie Institute. On p. 136 he gives the wave-length of the C line as 6399 instead of 6563. Such errors are of course of trivial importance in comparison with the general excellence of the book. Perhaps the greatest harm which could arise from them is that some immature reader might unjustly attribute to the author's experimental work a similar lack of accuracy and care.

HENRY G. GALE

Atlas typischer Spektren. BY J. M. EDER AND E. VALENTA. 53 charts with explanatory text of 143 pages. Published by the Komitee zur Verwaltung der Erbchaft Treitl, under the direction of the Kaiserliche Akademie der Wissenschaften. Wien, 1911. M. 78.

A collection of charts of spectra such as that offered by Eder and Valenta is an exacting task if the work is to take its place as a thoroughly efficient aid to the worker with spectra, be he physicist, chemist, or astronomer. It requires an extensive and highly flexible instrumental equipment, both as to light-sources and apparatus for photography of

the spectrum, a wide spectroscopic experience and high photographic technique on the part of the authors, and the utmost limit of the engraver's skill if a reasonable amount of the extraordinarily fine detail on the original negatives is to be reproduced. The work under review may be said to go far toward fulfilling each of these requirements.

Considering first the reproductions of spectra, we find the charts contain the flame spectra of 70 elements and compounds, the arc spectra of 78, and the spark spectra of 78. The total number of reproductions is about 640, the strips of spectra being arranged on 53 sheets. The authors wisely decided to have the plates made directly from the negatives, thus avoiding the intermediate positive which would mean the loss of much detail. Although the appearance is thus that of absorption spectra, no confusion can arise from this cause and in regular work there is some gain in having the chart similar in appearance to the original negative with which it is compared. The quality of the reproductions is probably as good as the engraving processes of the present day will permit. A scale beside each strip of spectrum gives intervals of 100 Å; while the wave-lengths of distinctive lines throughout the spectrum have their wave-lengths etched opposite them, impurity lines also often being indicated in this way.

An examination of the index shows that there are few chemical elements whose spectra are not presented in this work. A feature is the rich collection of spectra of the rare earths and of elements unknown a decade ago. The spectra are "typical" in the sense that they are given by the flame, arc, or spark with such instrumental arrangements as would generally be employed in laboratories, avoiding, except in a few cases, those conditions of the light-source which profoundly modify the character of the spectrum. The large collection of flame spectra will be of especial interest to chemists. In a number of cases the flame spectra of several compounds of the same element are presented. As an example may be mentioned the beautiful flutings of the chloride, bromide, iodide, and nitrate of copper, the scale being sufficient to show the distinctive differences in the arrangement of bands without attempting full resolution.

The first order of a concave grating of 146 cm radius was employed for most of the arc and spark spectra and for a few of the flame spectra. The scale obtained is about 11.6 Å per mm. The grating spectra are regularly reproduced in two portions, one from λ 2000 to λ 4600, and a second from λ 4300 to λ 7000. A few charts show the red region as far

as λ 8000. For a number of elements, besides grating spectra for the arc and spark, there are given prismatic spectra from two instruments, the one with glass prism showing the spectrum from λ 3500 to λ 7000, the scale at λ 4300 being about equal to that of the grating spectrum, the other with quartz prism showing the ultra-violet to the limits of transmission for air at about λ 1800. While there is some duplication in presenting both grating and prismatic spectra for the same element, the authors consider that the spectra obtained with a prism will be especially useful to those working with similar apparatus.

The scale of the reproductions is in general very satisfactory for spectra having a moderate number of lines. When, however, this scale is used for spectra whose stronger lines run into the hundreds and even thousands, little more than a general view of the distribution of lines is obtained from the charts. To have reproduced these spectra on a scale suitable for detailed study, say at least as large as 2 Å per mm, would have made the publication of prohibitive size, and this limitation was doubtless regretted by the authors as much as it can be by any user. The field is still open for a set of charts which will do full justice to these many lined spectra.

The volume of explanatory text deserves special notice. Flame, arc, and spark spectra are treated separately. For each substance the method of producing the radiation is briefly described, with many practical suggestions as to how certain features of the spectrum may best be brought out. Numerous references to original works are given. A table of wave-lengths follows for those lines which are distinct on the charts for the substance under discussion. These tables are a very valuable feature of the work, covering the spectrum in some cases between the extreme limits λ 2000 to λ 8000. The wave-lengths to 0.01 Å are compiled from the best available measurements. The result is the most complete and up-to-date collection of tables as regards number of spectra and range of wave-length which is at present available. While completeness in the individual spectra is not aimed at, the extent to which the distinctive lines are listed may be judged from the fact that the table for the iron arc includes 934 lines, that for the thorium arc 708 lines.

A regular reference to this atlas with its accompanying tables will greatly facilitate the work of anyone dealing with spectra. A few of those who use it will appreciate the enormous labor which Professors Eder and Valenta have expended in the compilation.

ARTHUR S. KING

EDITORIAL NOTE

Hereafter the duties of managing editor of the *Astrophysical Journal*, which have been successively borne by Mr. Hale and by Mr. Frost, will be assumed by Henry G. Gale, of the Department of Physics of the University of Chicago.

Manuscripts, proof sheets, books for review, and all editorial correspondence should henceforth be addressed to

EDITORS OF THE ASTROPHYSICAL JOURNAL
UNIVERSITY OF CHICAGO, CHICAGO, ILL.

ERRATUM

Vol. **34**, November 1911, in J. G. Hagen's article on "Various Scales for Color-Estimates": Page 267, line 4, *for* Innes *read* See.

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